

Introduction to Representations of Real Semisimple Lie Groups

Matvei Libine

Abstract

These are lecture notes for a one semester introductory course I gave at Indiana University. The goal was to make this exposition as clear and elementary as possible. A particular emphasis is given on examples involving $SU(1, 1)$. These notes are in part based on lectures given by my graduate advisor Wilfried Schmid at Harvard University and PQR2003 Euroschool in Brussels [15] as well as other sources, such as [9, 12, 13, 20, 22, 24]. The text is formatted for convenient viewing from iPad and other tablets.

Contents

1	Introduction	4
2	Some Geometry and Examples of Representations	7
2.1	Complex Projective Line	7
2.2	Examples of Representations	10
3	The Universal Enveloping Algebra	14
3.1	The Definition	14
3.2	Some Properties of the Universal Enveloping Algebra	17
3.3	Geometric Realization of $\mathcal{U}(\mathfrak{g})$	20
4	Irreducible Representations of $\mathfrak{sl}(2, \mathbb{R})$	21
4.1	Preliminaries	22
4.2	Classification of Irreducible $\mathfrak{sl}(2, \mathbb{C})$ -Modules	27

5	The Complexification, the Cartan Decomposition and Maximal Compact Subgroups	36
5.1	Assumptions on Groups	36
5.2	Complexifications and Real Forms	38
5.3	The Cartan Decomposition	39
5.4	Maximal Compact Subgroups	44
6	Definition of a Representation	46
6.1	A Few Words on Topological Vector Spaces . . .	47
6.2	The Definition	50
6.3	Admissible Representations	54
7	Harish-Chandra Modules	54
7.1	$K_{\mathbb{R}}$ -finite and \mathcal{C}^{∞} Vectors	55
7.2	Proof of Theorem 58	57
7.3	Harish-Chandra Modules	64
7.4	Infinitesimal Equivalence	70
7.5	A Few Words about Globalization	73
7.6	Unitary Representations and Infinitesimal Equivalence	75
8	Construction of Representations of $SU(1, 1)$	77
8.1	Construction of Representations of $SL(2, \mathbb{R})$. .	77
8.2	Construction of Representations of $SU(1, 1)$. .	78
8.3	Action of the Lie Algebra	82
8.4	The Harish-Chandra Module of $(\pi_{\chi}, \mathcal{C}^{\infty}(S^1))$. .	84
8.5	Decomposition of $(\pi_{\chi}, \mathcal{C}^{\infty}(S^1))$ into Irreducible Components	85
8.6	Unitary Representations of $SU(1, 1)$	89

9	Notes on Schur's Lemma	90
9.1	Classical Version	90
9.2	Schur's Lemma for Unitary Representations . .	91
9.3	Dixmier's Lemma	92
9.4	Infinitesimal Characters	94
10	Iwasawa and KAK Decompositions	95
10.1	Iwasawa Decomposition	95
10.2	KAK Decomposition	101
11	Principal Series Representations	102
11.1	$G_{\mathbb{R}}$ -equivariant Vector Bundles	102
11.2	Construction of the Principal Series Representations	103
11.3	Admissibility of the Principal Series Representations	107
11.4	Subrepresentation Theorem	109
11.5	Unitary Principal Series Representations	110
12	Group Characters	112
12.1	Hilbert-Schmidt Operators	112
12.2	Trace Class Operators	114
12.3	The Definition of a Character	116
12.4	Properties of Characters	119
12.5	Harish-Chandra Regularity Theorem	122
12.6	Characters of Representations of $SU(1, 1)$. . .	125

1 Introduction

Let G be a real semisimple or reductive Lie group (intuitively this means \mathfrak{g} is semisimple or reductive).

Examples: $SL(n, \mathbb{R})$, $GL(n, \mathbb{R})$, $O(p, q)$, $SU(p, q)$, $U(p, q)$, \dots
 \hat{G} = set of isomorphism classes of irreducible unitary representations. Thus, for each $i \in \hat{G}$, we get an irreducible unitary representation (π_i, V_i) .

G – compact: $\dim V_i < \infty$, unitarity is automatic (every irreducible representation is unitary), π_i is real analytic, so get a representation of \mathfrak{g} on V_i .

Non-compact: Typically $\dim V_i = \infty$, there are many interesting non-unitary representations, analyticity fails miserably. To get around this problem and construct representation of \mathfrak{g} , look at the space of \mathcal{C}^∞ or \mathcal{C}^ω vectors; solution: Harish-Chandra modules.

G – compact: We have Peter-Weyl Theorem:

$$L^2(G) \simeq \widehat{\bigoplus_{i \in \hat{G}} V_i \otimes V_i^*} \simeq \widehat{\bigoplus_{i \in \hat{G}} \text{End}(V_i)}$$

$G \times G$ equivariant isomorphism of vector spaces, the hat denotes the Hilbert space direct sum. We also have a Fourier Inversion Theorem: Define

$$\mathcal{C}^\infty(G) \ni f \mapsto \hat{f}(i) = \int_G f(g) \cdot \pi_i(g) dg \in \text{End}(V_i),$$

then

$$f(e) = \sum_{i \in \hat{G}} \text{Tr } \hat{f}(i) \cdot \dim V_i.$$

Compare this with Fourier series:

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \cdot e^{inx}, \quad f(0) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx, \quad f \in \mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z}).$$

Non-compact: Not every irreducible unitary representation occurs in $L^2(G)$. In fact, it is still an open problem to describe \hat{G} for an arbitrary real semisimple Lie group. We have an abstract Fourier Inversion Theorem:

$$L^2(G) \simeq \int_{i \in \hat{G}} V_i \hat{\otimes} V_i^* d\mu(i) \simeq \int_{i \in \hat{G}} \text{End}_{HS}(V_i) d\mu(i),$$

where $\mu(i)$ denotes a measure on \hat{G} , called Plancherel measure. For compact G , $\mu(i) = \dim V_i$ and $\text{supp}(\mu) = \hat{G}$. For non-compact G , $\text{supp}(\mu) \subsetneq \hat{G}$.

$$f(e) = \int_{\hat{G}} \text{Tr} \hat{f}(i) d\mu(i), \quad f \in \mathcal{C}_c^\infty(G).$$

G – compact: We have Borel-Weil-Bott Theorem for $H^i(G_{\mathbb{C}}/B_{\mathbb{C}}, \mathcal{L}_\lambda)$, irreducible representations are uniquely determined by their characters, Weyl and Kirillov's character formulas.

Non-compact: Naive notion of character fails. Harish-Chandra: get *character* as a conjugation-invariant distribution on G , does determine representation up to equivalence. Regularity theorem: characters are real-analytic functions with singularities. No totally explicit formula for characters. Kashiwara-Schmid (~ 1994) constructed representations of G in $H^i(G_{\mathbb{C}}/B_{\mathbb{C}}, \mathcal{F})$,

where \mathcal{F} is a G -equivariant sheaf on $G_{\mathbb{C}}/B_{\mathbb{C}}$. This can be regarded as an analogue of Borel-Weil-Bott theorem. Schmid-Vilonen (~ 1998) gave analogues of Weil and Kirillov's character formulas.

G – compact: If $H \subset G$ is a closed subgroup,

$$L^2(G/H) \hookrightarrow L^2(G).$$

Non-compact: If $H \subset G$ is a closed non-compact subgroup,

$$L^2(G/H) \not\hookrightarrow L^2(G).$$

For example, if $G = SL(2, \mathbb{R})$, $\Gamma \subset G$ a discrete subgroup, even then $L^2(G/\Gamma)$ is not known in general.

Strategy for understanding representations: In order to understand representations of compact groups K we restricted them to a maximal torus T , and the representation theory for T is trivial. Equivalently, in order to understand finite-dimensional representations of a semisimple Lie algebra \mathfrak{k} we restrict them to a Cartan subalgebra \mathfrak{h} . Now, to understand representations of a real semisimple Lie group G we restrict them to a maximal compact subgroup K , and the representation theory for K is assumed to be understood. (A particular choice of a maximal compact subgroup $K \subset G$ does not matter, just like a particular choice of a maximal torus $T \subset K$ or a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$ does not matter.)

2 Some Geometry and Examples of Representations

2.1 Complex Projective Line

The complex projective line, usually denoted by $\mathbb{C}P^1$ or $\mathbb{P}^1(\mathbb{C})$, is $\mathbb{C}^2 \setminus \{0\}$ modulo equivalence, where

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim \begin{pmatrix} \lambda z_1 \\ \lambda z_2 \end{pmatrix}, \quad \forall \lambda \in \mathbb{C}^\times.$$

Note that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim \begin{pmatrix} z_1/z_2 \\ 1 \end{pmatrix} \text{ if } z_2 \neq 0, \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ z_2/z_1 \end{pmatrix} \text{ if } z_2 \neq 0.$$

We identify points $\begin{pmatrix} z \\ 1 \end{pmatrix} \in \mathbb{C}P^1$ with $z \in \mathbb{C}$, then the complement of \mathbb{C} in $\mathbb{C}P^1$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and we think of this point as ∞ .

Thus

$$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \approx S^2 \quad (\text{as } \mathcal{C}^\infty \text{ manifolds}).$$

The group $SL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix},$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az + b \\ cz + d \end{pmatrix} \sim \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}.$$

The map $z \mapsto \frac{az+b}{cz+d}$ from $\mathbb{C} \cup \{\infty\}$ to itself is also called a Möbius or conformal transformation. Note that the center $\{\pm Id\} \subset SL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ trivially, so we actually have action on $\mathbb{C}P^1$ of

$$PGL(2, \mathbb{C}) = GL(2, \mathbb{C}) / \{\text{center}\} = SL(2, \mathbb{C}) / \{\pm Id\}.$$

Recall that an *automorphism* of \mathbb{CP}^1 or any complex manifold M is a complex analytic map $M \rightarrow M$ which has an inverse that is also complex analytic.

Theorem 1. *All automorphisms of \mathbb{CP}^1 are of the form*

$$z \mapsto \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Moreover, two matrices $M_1, M_2 \in SL(2, \mathbb{C})$ produce the same automorphism on \mathbb{CP}^1 if and only if $M_2 = \pm M_1$.

Next we turn our attention to the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}.$$

Note that $SL(2, \mathbb{R})$ preserves \mathbb{H} : If $a, b, c, d \in \mathbb{R}$ and $\operatorname{Im} z > 0$, then

$$\begin{aligned} \operatorname{Im} \left(\frac{az + b}{cz + d} \right) &= \frac{1}{2i} \left(\frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d} \right) \\ &= \frac{1}{2i} \frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{1}{2i} \frac{(ad - bc)(z - \bar{z})}{|cz + d|^2} = \frac{\operatorname{Im} z}{|cz + d|^2} > 0. \end{aligned}$$

Thus the group $SL(2, \mathbb{R})$ acts on \mathbb{H} by automorphisms.

Theorem 2. *All automorphisms of the upper half-plane \mathbb{H} are of the form*

$$z \mapsto \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

Moreover, two matrices $M_1, M_2 \in SL(2, \mathbb{R})$ produce the same automorphism on \mathbb{CP}^1 if and only if $M_2 = \pm M_1$.

The group $PGL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm Id\}$ is also the automorphism group of the lower half-plane $\bar{\mathbb{H}} = \{z \in \mathbb{C}; \text{Im } z < 0\}$ and preserves $\mathbb{R} \cup \{\infty\}$.

Note that the Cayley transform

$$z \mapsto \frac{z - i}{z + i},$$

which is the fractional linear transformation associated to the matrix $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, takes $\bar{\mathbb{H}}$ to the unit disk

$$\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}.$$

Of course, the group of automorphisms of \mathbb{D} is obtained from the group of automorphisms of $\bar{\mathbb{H}}$ by conjugating by C :

$$C \cdot PGL(2, \mathbb{R}) \cdot C^{-1} \subset PGL(2, \mathbb{C}).$$

Lemma 3. *We have $C \cdot SL(2, \mathbb{R}) \cdot C^{-1} = SU(1, 1)$, where*

$$SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SL(2, \mathbb{C}); a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

In particular, the groups $SL(2, \mathbb{R})$ and $SU(1, 1)$ are isomorphic.

We immediately obtain:

Theorem 4. *All automorphisms of the unit disk \mathbb{D} are of the form*

$$z \mapsto \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1, 1).$$

Moreover, two matrices $M_1, M_2 \in SU(1, 1)$ produce the same automorphism on \mathbb{CP}^1 if and only if $M_2 = \pm M_1$.

The group $SU(1, 1)/\{\pm Id\}$ is also the automorphism group of $\{z \in \mathbb{C}; |z| > 1\} \cup \{\infty\}$ and preserves the unit circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$.

2.2 Examples of Representations

In general, if G is any kind of group acting on a set X , then we automatically get a representation π of G in the vector space V consisting of complex (or real) valued functions on X :

$$(\pi(g)f)(x) = f(g^{-1} \cdot x), \quad g \in G, f \in V, x \in X.$$

Note that the inverse in $f(g^{-1} \cdot x)$ ensures $\pi(g_1)\pi(g_2) = \pi(g_1g_2)$, without it we would have $\pi(g_1)\pi(g_2) = \pi(g_2g_1)$.

Now, let

$$V = \{\text{holomorphic functions on the upper half-plane } \mathbb{H}\},$$

clearly, V is a vector space over \mathbb{C} of infinite dimension. Define a representation of $SL(2, \mathbb{R})$ on V by

$$(\pi(g)f)(z) = f(g^{-1} \cdot z), \quad g \in SL(2, \mathbb{R}), f \in V, z \in \mathbb{H}.$$

Explicitly, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and

$$(\pi(g)f)(z) = f(g^{-1} \cdot z) = f\left(\frac{dz - b}{-cz + a}\right).$$

Note that V is not irreducible – it contains a subrepresentation V_0 spanned by the constant functions, which form the one-dimensional trivial representation. On the other hand, V is indecomposable, i.e. cannot be written as a direct sum of two subrepresentations (this part is harder to see). Once we learn

about Harish-Chandra modules, it will be easy to see that V_0 does not have an $SL(2, \mathbb{R})$ -invariant direct sum complement, in fact, V_0 is the only closed invariant subspace of V , and the quotient representation on V/V_0 is irreducible. Note also that (π, V) is *not* unitary (if it were unitary, then one could take the orthogonal complement of V_0 and write V as a direct sum of subrepresentations $V_0 \oplus V_0^\perp$).

This example is in many ways typical to representations of real semisimple Lie groups: the vector spaces are infinite-dimensional and have some sort of topological structure, the representations may not have unitary structure, need not decompose into direct sum of irreducible subrepresentations and may have interesting components appearing as quotients. This is quite different from the representations of compact Lie groups.

Let us modify the last example by letting

$$V = \{\text{holomorphic functions on the unit disk } \mathbb{D}\},$$

and defining a representation of $SU(1, 1)$ on V by

$$(\pi(g)f)(z) = f(g^{-1} \cdot z), \quad g \in SU(1, 1), \quad f \in V, \quad z \in \mathbb{D}.$$

Explicitly, if $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, then $g^{-1} = \begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix}$, and

$$(\pi(g)f)(z) = f(g^{-1} \cdot z) = f\left(\frac{\bar{a}z - b}{-\bar{b}z + a}\right).$$

Of course, this is the same example of representation as before. However, this $SU(1, 1)$ acting on the unit disk \mathbb{D} model has a certain advantage. We already mentioned that the strategy for understanding representations of real semisimple Lie groups is by reduction to maximal compact subgroup. In the case of

$SU(1, 1)$, a maximal compact subgroup can be chosen to be the subgroup of diagonal matrices

$$K = \left\{ k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}; \theta \in [0, 2\pi) \right\} \subset SU(1, 1).$$

In the case of $SL(2, \mathbb{R})$, a maximal compact subgroup can be chosen to be the subgroup of rotation matrices

$$K' = \left\{ k'_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}; \theta \in [0, 2\pi) \right\} \subset SL(2, \mathbb{R}).$$

The action of k_θ is particularly easy to describe:

$$k_\theta \cdot z = e^{2i\theta} z, \quad (\pi(k_\theta)f)(z) = f(e^{-2i\theta} z)$$

– these are rotations by 2θ . On the other hand, the action of k'_θ is much harder to visualize:

$$k'_\theta \cdot z = \frac{\cos \theta z - \sin \theta}{\sin \theta z + \cos \theta}, \quad (\pi(k'_\theta)f)(z) = f\left(\frac{\cos \theta z + \sin \theta}{-\sin \theta z + \cos \theta}\right),$$

but one can identify K' with the isotropy subgroup of $i \in \mathbb{H}$.

Let us find eigenvectors in

$$V = \{\text{holomorphic functions on the unit disk } \mathbb{D}\}$$

with respect to the action of K . These are functions z^m , $m = 0, 1, 2, 3, \dots$

$$(\pi(k_\theta)z^m)(z) = (e^{-2i\theta} z)^m = e^{-2mi\theta} \cdot z^m.$$

The algebraic direct sum of eigenspaces $\bigoplus_{m \geq 0} \mathbb{C} \cdot z^m$ is just the subspace of polynomial functions on \mathbb{C} . These are dense in V , and we express that by writing

$$V = \widehat{\bigoplus_{m \geq 0} \mathbb{C} \cdot z^m}.$$

Later we will see that $\bigoplus_{m \geq 0} \mathbb{C} \cdot z^m$ is the underlying Harish-Chandra module of V .

Fix an integer $n \geq 0$ and let

$$V = \{\text{holomorphic functions on the unit disk } \mathbb{D}\}.$$

Define a representation of $SU(1, 1)$ by

$$\begin{aligned} (\pi_n(g)f)(z) &= (-\bar{b}z + a)^{-n} \cdot f\left(\frac{\bar{a}z - b}{-\bar{b}z + a}\right), \\ g &= \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1, 1), \quad f \in V, \quad z \in \mathbb{D}. \end{aligned}$$

You can verify that $\pi_n(g_1)\pi_n(g_2) = \pi_n(g_1g_2)$ is true, so we get a representation. If $n = 0$ we get the same representation as before. If $n \geq 2$ we get the so-called *holomorphic discrete series* representation, and if $n = 1$ we get the so-called *limit of the holomorphic discrete series* representation. To get the so-called *antiholomorphic discrete series and its limit* representations one uses the antiholomorphic functions on \mathbb{D} .

Finally, we describe the (not necessarily unitary) *principal series* representations of $SU(1, 1)$. These require a parameter $\lambda \in \mathbb{C}$. Let $S^1 = \{z \in \mathbb{C}; |z| = 1\}$ be the unit circle and define

$$\tilde{V} = \{\text{smooth functions on the unit circle } S^1\},$$

$$\begin{aligned} (\pi_\lambda^+(g)f)(z) &= |-\bar{b}z + a|^{-1-\lambda} \cdot f\left(\frac{\bar{a}z - b}{-\bar{b}z + a}\right), \\ g &= \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1, 1), \quad f \in \tilde{V}, \quad z \in S^1. \end{aligned}$$

It is still true that $\pi_\lambda^+(g_1)\pi_\lambda^+(g_2) = \pi_\lambda^+(g_1g_2)$, so we get a representation. Note that $\pi_\lambda^+(-Id) = Id_{\tilde{V}}$, there is another principal series with the property $\pi_\lambda^-(-Id) = -Id_{\tilde{V}}$.

Later we will construct the principal series representations for every real semisimple Lie group G . These representations are very important because, in a certain sense, every representation of G occurs as a subrepresentation of some principal series representation.

3 The Universal Enveloping Algebra

In this section, let \mathbb{k} be any field (not necessarily \mathbb{R} or \mathbb{C}) and let \mathfrak{g} be a Lie algebra over \mathbb{k} .

3.1 The Definition

We would like to embed \mathfrak{g} into a (large) associative algebra \mathcal{U} so that

$$[X, Y] = XY - YX, \quad \forall X, Y \in \mathfrak{g},$$

and \mathcal{U} is “as close to the free associative algebra as possible”. First we consider the tensor algebra

$$\bigotimes \mathfrak{g} = \bigoplus_{n=0}^{\infty} (\otimes^n \mathfrak{g}), \quad \otimes^n \mathfrak{g} = \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n \text{ times}}, \quad \otimes^0 \mathfrak{g} = \mathbb{k}.$$

Then $\bigotimes \mathfrak{g}$ is an algebra over \mathbb{k} which is associative, non-commutative, generated by \mathfrak{g} and has a unit. It is universal in the sense that any other \mathbb{k} -algebra with those properties is a quotient of $\bigotimes \mathfrak{g}$.

Similarly, we can define

$$S(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g}),$$

where $S^0(\mathfrak{g}) = \mathbb{k}$ and

$$S^n(\mathfrak{g}) = n^{\text{th}} \text{ symmetric power of } \mathfrak{g}.$$

In other words, $S(\mathfrak{g})$ is the algebra of polynomials over \mathfrak{g}^* and $S^n(\mathfrak{g})$ is the subspace of homogeneous polynomials over \mathfrak{g}^* of homogeneity degree n , where \mathfrak{g}^* is the vector space dual of \mathfrak{g} . Note that $S(\mathfrak{g})$ contains \mathfrak{g} as a linear subspace and is generated by it. Then $S(\mathfrak{g})$ is an algebra over \mathbb{k} which is associative, commutative, generated by \mathfrak{g} and has unit. It is universal in the sense that any other \mathbb{k} -algebra with those properties is a quotient of $S(\mathfrak{g})$. The algebra $S(\mathfrak{g})$ can be realized as a quotient of $\bigotimes \mathfrak{g}$:

$$S(\mathfrak{g}) = \bigotimes \mathfrak{g} / I$$

where I is the two-sided ideal generated by $XY - YX$, $\forall X, Y \in \mathfrak{g}$. If $\text{char } \mathbb{k} = 0$, we can also realize $S(\mathfrak{g})$ as a subalgebra of $\bigotimes \mathfrak{g}$. By construction, both $\bigotimes \mathfrak{g}$ and $S(\mathfrak{g})$ are graded algebras:

$$\begin{aligned} \bigotimes \mathfrak{g} &= \bigoplus_{n=0}^{\infty} (\bigotimes^n \mathfrak{g}), & (\bigotimes^m \mathfrak{g}) \cdot (\bigotimes^n \mathfrak{g}) &\subset \bigotimes^{m+n} \mathfrak{g}, \\ S(\mathfrak{g}) &= \bigoplus_{n=0}^{\infty} S^n(\mathfrak{g}), & S^m(\mathfrak{g}) \cdot S^n(\mathfrak{g}) &\subset S^{m+n}(\mathfrak{g}). \end{aligned}$$

Definition 5. *The universal enveloping algebra of \mathfrak{g} is*

$$\mathcal{U}(\mathfrak{g}) = \bigotimes \mathfrak{g} / J,$$

where J is the two-sided ideal generated by

$$XY - YX - [X, Y], \quad \forall X, Y \in \mathfrak{g}.$$

Then $\mathcal{U}(\mathfrak{g})$ is an associative algebra over \mathbb{k} with unit equipped with a \mathbb{k} -linear map $i : \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$ such that

1. i is a Lie algebra homomorphism, where $\mathcal{U}(\mathfrak{g})$ is given structure of a Lie algebra by $[a, b] =_{\text{def}} ab - ba$;
2. $\mathcal{U}(\mathfrak{g})$ is generated by $i(\mathfrak{g})$ as a \mathbb{k} -algebra.

We will see soon that $i : \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$ is an injection, but it is not clear at this point.

Caution: Let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$, then $E^2 = 0$ in the sense of multiplication of 2×2 matrices, but $E^2 \neq 0$ as an element of $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$.

The algebra $\mathcal{U}(\mathfrak{g})$ is *universal* in the following sense:

Theorem 6. *Let A be any associative algebra over \mathbb{k} with unit, and let $\rho : \mathfrak{g} \rightarrow A$ be a \mathbb{k} -linear map such that*

$$\rho(X)\rho(Y) - \rho(Y)\rho(X) = \rho([X, Y]), \quad \forall X, Y \in \mathfrak{g}.$$

Then ρ can be uniquely extended to a morphism of associative algebras with unit $\rho : \mathcal{U}(\mathfrak{g}) \rightarrow A$.

Corollary 7. *Any representation of \mathfrak{g} (not necessarily finite-dimensional) has a canonical structure of a $\mathcal{U}(\mathfrak{g})$ -module. Conversely, every $\mathcal{U}(\mathfrak{g})$ -module has a canonical structure of a representation of \mathfrak{g} .*

Restatement: The categories of representations of \mathfrak{g} and $\mathcal{U}(\mathfrak{g})$ -modules are equivalent.

3.2 Some Properties of the Universal Enveloping Algebra

Note that $\mathcal{U}(\mathfrak{g})$ is a *filtered* algebra: Let

$$\mathcal{U}_n(\mathfrak{g}) =_{\text{def}} \text{image of } \bigoplus_{k=0}^n (\otimes^k \mathfrak{g}) \text{ in } \mathcal{U}(\mathfrak{g}) = \bigotimes \mathfrak{g} / J,$$

then

$$\mathbb{k} = \mathcal{U}_0(\mathfrak{g}) \subset \mathcal{U}_1(\mathfrak{g}) \subset \cdots \subset \mathcal{U}_n(\mathfrak{g}) \subset \cdots \subset \mathcal{U}(\mathfrak{g}) = \bigcup_{n=0}^{\infty} \mathcal{U}_n(\mathfrak{g})$$

$$\text{and } \mathcal{U}_m(\mathfrak{g}) \cdot \mathcal{U}_n(\mathfrak{g}) \subset \mathcal{U}_{m+n}(\mathfrak{g}).$$

Caution: $\mathcal{U}(\mathfrak{g})$ is *not* a graded algebra.

Proposition 8. 1. If $X \in \mathcal{U}_m(\mathfrak{g})$ and $Y \in \mathcal{U}_n(\mathfrak{g})$, then

$$XY - YX \in \mathcal{U}_{m+n-1}(\mathfrak{g}).$$

2. Let $\{X_1, \dots, X_r\}$ be an ordered basis of \mathfrak{g} , the monomials

$$(X_1)^{k_1} \cdot (X_2)^{k_2} \cdot \dots \cdot (X_r)^{k_r}, \quad \sum_{i=1}^r k_i \leq n, \quad \text{span } \mathcal{U}_n(\mathfrak{g}).$$

(Note that we have fixed the order of basis elements.)

Proof. We prove the first part by induction on m . So suppose first that $m = 1$ and $X \in i(\mathfrak{g})$, then

$$\begin{aligned} & XY_1 \dots Y_n - Y_1 \dots Y_n X \\ &= XY_1 Y_2 \dots Y_n - Y_1 X Y_2 \dots Y_n \\ &+ Y_1 X Y_2 \dots Y_n - Y_1 Y_2 X Y_3 \dots Y_n \\ &\quad \dots \\ &+ Y_1 \dots Y_{n-1} X Y_n - Y_1 \dots Y_n X \\ &= \sum_{i=1}^n Y_1 \dots [X, Y_i] \dots Y_n \in \mathcal{U}_n(\mathfrak{g}). \end{aligned}$$

This implies if $X \in \mathcal{U}_1(\mathfrak{g})$, $Y \in \mathcal{U}_n(\mathfrak{g})$, then $[X, Y] \in \mathcal{U}_n(\mathfrak{g})$, i.e.

$$XY \equiv YX \pmod{\mathcal{U}_n(\mathfrak{g})}.$$

Then

$$\begin{aligned} X_1 \dots X_m X_{m+1} Y &\equiv X_1 \dots X_m Y X_{m+1} \\ &\equiv Y X_1 \dots X_m X_{m+1} \pmod{\mathcal{U}_{m+n}(\mathfrak{g})} \end{aligned}$$

and the first part follows.

To prove the second part we also use induction on n . If $n = 1$ the statement amounts to

$$\mathcal{U}_1(\mathfrak{g}) = \mathbb{k}\text{-Span}\{1, X_1, \dots, X_r\},$$

which is true. Note that $\mathcal{U}_{n+1}(\mathfrak{g}) = \mathcal{U}_1(\mathfrak{g}) \cdot \mathcal{U}_n(\mathfrak{g})$. By induction hypothesis,

$$\mathcal{U}_n(\mathfrak{g}) = \mathbb{k}\text{-Span}\left\{(X_1)^{k_1} \cdot (X_2)^{k_2} \cdot \dots \cdot (X_r)^{k_r}; \sum_{i=1}^r k_i \leq n\right\}.$$

By the first part,

$$X_j \cdot (X_1)^{k_1} \cdot \dots \cdot (X_r)^{k_r} - (X_1)^{k_1} \cdot \dots \cdot (X_j)^{k_j+1} \cdot \dots \cdot (X_r)^{k_r}$$

lies in $\mathcal{U}_n(\mathfrak{g})$. Hence $X_j \cdot (X_1)^{k_1} \cdot \dots \cdot (X_r)^{k_r}$ lies in

$$\mathbb{k}\text{-Span}\left\{(X_1)^{k_1} \cdot (X_2)^{k_2} \cdot \dots \cdot (X_r)^{k_r}; \sum_{i=1}^r k_i \leq n+1\right\}.$$

□

Corollary 9. *The associated graded algebra*

$$Gr\mathcal{U}(\mathfrak{g}) =_{def} \bigoplus_{n=0}^{\infty} \mathcal{U}_n(\mathfrak{g}) / \mathcal{U}_{n-1}(\mathfrak{g})$$

is commutative.

Since $S(\mathfrak{g})$ is a universal associative commutative algebra, we get a unique map $S(\mathfrak{g}) \rightarrow \text{Gr}\mathcal{U}(\mathfrak{g})$ such that

$$S(\mathfrak{g}) \ni X \mapsto i(X) \in \text{Gr}\mathcal{U}(\mathfrak{g}), \quad \forall X \in \mathfrak{g}.$$

Theorem 10 (Poincaré-Birkhoff-Witt). *This map $S(\mathfrak{g}) \rightarrow \text{Gr}\mathcal{U}(\mathfrak{g})$ is an isomorphism of algebras.*

The Poincaré-Birkhoff-Witt Theorem is valid over *any* field \mathbb{k} , even if $\text{char } \mathbb{k} \neq 0$.

Corollary 11. *The ordered monomials*

$$\left\{ (X_1)^{k_1} \cdot (X_2)^{k_2} \cdot \dots \cdot (X_r)^{k_r}; \sum_{i=1}^r k_i \leq n \right\}$$

form a vector space basis of $\mathcal{U}_n(\mathfrak{g})$.

Corollary 12. *The map $i : \mathfrak{g} \rightarrow \mathcal{U}(\mathfrak{g})$ is injective, hence \mathfrak{g} can be regarded as a vector subspace of $\mathcal{U}(\mathfrak{g})$.*

Corollary 13. *If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, then the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces $\mathcal{U}(\mathfrak{h}) \hookrightarrow \mathcal{U}(\mathfrak{g})$. Moreover, $\mathcal{U}(\mathfrak{g})$ is free as left $\mathcal{U}(\mathfrak{h})$ -module.*

Corollary 14. *If $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ are Lie subalgebras such that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ as vector spaces (\mathfrak{h}_1 and \mathfrak{h}_2 need not be ideals), then the multiplication map $\mathcal{U}(\mathfrak{h}_1) \otimes \mathcal{U}(\mathfrak{h}_2) \rightarrow \mathcal{U}(\mathfrak{g})$ is a vector space isomorphism.*

Corollary 15. *$\mathcal{U}(\mathfrak{g})$ has no zero divisors.*

Example 16. *Let $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C})$, then $E^n \neq 0$ in $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$ for all n .*

Proposition 17. Assume that $\text{char } \mathbb{k} = 0$, then the map $S(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$ defined on monomials by

$$\text{Sym}(X_1 \dots X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \dots X_{\sigma(n)}$$

is an isomorphism of \mathfrak{g} -modules.

Caution: This is *not* an algebra isomorphism unless \mathfrak{g} is commutative.

3.3 Geometric Realization of $\mathcal{U}(\mathfrak{g})$

We now assume that the field \mathbb{k} is \mathbb{R} or \mathbb{C} . Just as Lie algebra \mathfrak{g} can be identified with left-invariant vector fields on G , the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ can be identified with

$$\mathcal{D}^l = \{\text{left-invariant differential operators on } G\}.$$

Indeed, \mathcal{D}^l is an associative algebra over \mathbb{R} or \mathbb{C} with unit, it is filtered by the order of differential operators. We have a map $\mathfrak{g} \rightarrow \mathcal{D}_1^l$ sending an element of Lie algebra into the corresponding left-invariant vector field, which is a differential operator of order 1. By the universality of $\mathcal{U}(\mathfrak{g})$, we get an algebra homomorphism

$$j : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}^l$$

which respects the filtration: $j(\mathcal{U}_n(\mathfrak{g})) \subset \mathcal{D}_n^l$, $n = 0, 1, 2, 3, \dots$. Hence we get a map

$$j : \text{Gr}\mathcal{U}(\mathfrak{g}) \rightarrow \text{Gr}\mathcal{D}^l.$$

Note that for an operator $D \in \mathcal{D}_n^l$, its symbol $\sigma_n(D)$ is left-invariant under the action of G and hence uniquely determined

by the value at $e \in G$, so we get an injective map

$$\sigma : Gr\mathcal{D}^l \hookrightarrow S(T_e G) = S(\mathfrak{g})$$

which is an algebra homomorphism.

Let us consider the following composition of algebra homomorphisms

$$S(\mathfrak{g}) \rightarrow Gr\mathcal{U}(\mathfrak{g}) \xrightarrow{j} Gr\mathcal{D}^l \xrightarrow{\sigma} S(\mathfrak{g}).$$

This composition is the identity map, since it is the identity map for elements of degree 1, i.e. elements in \mathfrak{g} . Thus we obtain a commutative diagram:

$$\begin{array}{ccc} Gr\mathcal{U}(\mathfrak{g}) & \xrightarrow{j} & Gr\mathcal{D}^l \\ \text{onto} \uparrow & & \downarrow \sigma, \text{ injective} \\ S(\mathfrak{g}) & \equiv & S(\mathfrak{g}) \end{array}$$

Each map in the diagram must be an isomorphism. In particular, we have proved the Poincaré-Birkhoff-Witt Theorem when the field \mathbb{k} is \mathbb{R} or \mathbb{C} . We conclude:

Proposition 18. *We have an isomorphism of filtered algebras $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{D}^l$. Under this isomorphism, the center of $\mathcal{U}(\mathfrak{g})$ is identified with the subalgebra of bi-invariant differential operators on G .*

4 Irreducible Representations of $\mathfrak{sl}(2, \mathbb{R})$

In this section we classify all irreducible representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, these are the same as the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. The results of this section will be used to classify all irreducible representations of $SL(2, \mathbb{R})$.

4.1 Preliminaries

Recall that a *representation* of a Lie algebra \mathfrak{g} is a complex vector space V (possibly of infinite dimension) together with a map $\pi : \mathfrak{g} \rightarrow \text{End}(V)$ such that

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X), \quad \forall X, Y \in \mathfrak{g}.$$

Note that, unlike the Lie group representation, vector space V is not required to have any topology whatsoever. If \mathfrak{g} is a real Lie algebra, V may be taken a real vector space, but we prefer to work with complex vector spaces. Representations of \mathfrak{g} are often called *\mathfrak{g} -modules*.

When the Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$ or $\mathfrak{sl}(2, \mathbb{C})$, it is convenient to fix the following set of generators:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These generators satisfy the following relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

We classify all irreducible representations (π, V) of $\mathfrak{sl}(2, \mathbb{C})$ that satisfy the following additional assumption: *There exists a $\lambda \in \mathbb{C}$ such that the λ -eigenspace for H in V has dimension one, i.e.*

$$\pi(H)v_0 = \lambda v_0, \quad \text{for some } v_0 \in V, v_0 \neq 0, \text{ and } \lambda \in \mathbb{C}$$

and

$$\forall v \in V, \quad \pi(H)v = \lambda v \quad \implies \quad v \text{ is a scalar multiple of } v_0.$$

We will see soon that this assumption forces V to decompose into a direct sum of eigenspaces for H and each eigenspace has dimension one. The $\mathfrak{sl}(2, \mathbb{C})$ -modules associated to irreducible representations of $SL(2, \mathbb{R})$ and all irreducible finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -modules satisfy this assumption automatically.

Lemma 19. *Let $v \in V$ be an eigenvector for H with eigenvalue $\nu \in \mathbb{C}$, then*

1. $\pi(E)v \in V$ is either zero or an eigenvector for H with eigenvalue $\nu + 2$;
2. $\pi(F)v \in V$ is either zero or an eigenvector for H with eigenvalue $\nu - 2$.

Proof. We have:

$$\begin{aligned}\pi(H)(\pi(E)v) &= \pi([H, E])v + \pi(E)\pi(H)v \\ &= 2\pi(E)v + \nu\pi(E)v = (\nu + 2)\pi(E)v;\end{aligned}$$

$$\begin{aligned}\pi(H)(\pi(F)v) &= \pi([H, F])v + \pi(F)\pi(H)v \\ &= -2\pi(F)v + \nu\pi(F)v = (\nu - 2)\pi(F)v.\end{aligned}$$

□

Define

$$v_k = (\pi(E))^k v_0, \quad v_{-k} = (\pi(F))^k v_0, \quad k = 1, 2, 3, \dots$$

Then

$$\pi(E)v_k = v_{k+1} \quad \text{if } k \geq 0, \quad \pi(F)v_k = v_{k-1} \quad \text{if } k \leq 0.$$

Corollary 20. *Each vector $v_k \in V$, $k \in \mathbb{Z}$, is either zero or an eigenvector for H with eigenvalue $\lambda + 2k$.*

Let us consider the *Casimir element*

$$\Omega =_{\text{def}} H^2 + 2EF + 2FE \in \mathcal{U}(\mathfrak{sl}(2, \mathbb{C})).$$

This element is distinguished by the property that it generates $\mathcal{ZU}(\mathfrak{sl}(2, \mathbb{C}))$ – the center of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$. We will discuss its properties later.

Lemma 21. *Let $v_0 \in V$ be an eigenvector of H with eigenvalue λ such that $\dim_{\mathbb{C}} \ker(\pi(H) - \lambda) = 1$. Then*

$$\pi(\Omega)v_0 = \pi(H)^2v_0 + 2\pi(E)\pi(F)v_0 + 2\pi(F)\pi(E)v_0 = \mu v_0$$

for some $\mu \in \mathbb{C}$.

Proof. First of all, $\pi(H)^2v_0 = \lambda^2v_0$. By Lemma 19, $\pi(F)v_0$ is either zero or an eigenvector for H with eigenvalue $\lambda - 2$ and $\pi(E)\pi(F)v_0$ is either zero or an eigenvector for H with eigenvalue λ . Then our assumption $\dim_{\mathbb{C}} \ker(\pi(H) - \lambda) = 1$ implies $\pi(E)\pi(F)v_0$ is a scalar multiple of v_0 . Similarly, $\pi(F)\pi(E)v_0$ is a scalar multiple of v_0 . Therefore, $\pi(\Omega)v_0$ is a scalar multiple of v_0 . \square

Corollary 22. *Under the assumptions of the Lemma, $\pi(\Omega)v_k = \mu v_k$ for all $k \in \mathbb{Z}$.*

Proof. This is a manifestation of the fact that Ω is a central element of $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$. Hence $\pi(\Omega)\pi(X) = \pi(X)\pi(\Omega)$ for all $X \in \mathfrak{g}$. \square

Next we would like to prove that

$$V = \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \cdot v_k.$$

Since V is irreducible, it is sufficient to prove that the vector space

$$V_0 =_{\text{def}} \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \cdot v_k$$

is invariant under the action of $\mathfrak{sl}(2, \mathbb{C})$. Obviously, V_0 is invariant under H , and it would suffice to prove that $\pi(F)v_k$ is proportional to v_{k-1} for $k > 0$ and that $\pi(E)v_k$ is proportional to v_{k+1} for $k < 0$.

Lemma 23. *Let $v_0 \in V$ be an eigenvector for H with eigenvalue λ , define $V_0 = \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \cdot v_k$ and suppose that $\pi(\Omega)v_0 = \mu v_0$ for some $\mu \in \mathbb{C}$. Then the set of nonzero v_k forms a basis for V_0 with the following relations:*

$$\pi(H)v_k = (\lambda + 2k)v_k, \quad k \in \mathbb{Z},$$

$$\pi(E)v_k = v_{k+1} \quad \text{if } k \geq 0, \quad \pi(F)v_k = v_{k-1} \quad \text{if } k \leq 0,$$

$$\pi(E)v_k = \frac{1}{4}(\mu - (\lambda + 2k + 2)^2 + 2(\lambda + 2k + 2))v_{k+1} \quad \text{if } k < 0,$$

$$\pi(F)v_k = \frac{1}{4}(\mu - (\lambda + 2k - 2)^2 - 2(\lambda + 2k - 2))v_{k-1} \quad \text{if } k > 0.$$

In particular, V_0 is $\mathfrak{sl}(2, \mathbb{C})$ -invariant and all nontrivial H eigenspaces of V_0 are one-dimensional.

Proof. By Corollary 22, $\pi(\Omega)v = \mu v$ for all $v \in V_0$. We can rewrite Ω as

$$\Omega = H^2 + 2EF + 2FE = H^2 + 2H + 4FE = H^2 - 2H + 4EF.$$

To simplify notations, let us suppose that $u_\nu \in V_0$ is such that $\pi(H)u_\nu = \nu u_\nu$ for some $\nu \in \mathbb{C}$. Of course, we will let $u_\nu = v_k$ with $\nu = \lambda + 2k$, $k \in \mathbb{Z}$. Then

$$\pi(E)\pi(F)u_\nu = \frac{1}{4}(\pi(\Omega) - \pi(H)^2 + 2\pi(H))u_\nu = \frac{1}{4}(\mu - \nu^2 + 2\nu)u_\nu,$$

$$\pi(F)\pi(E)u_\nu = \frac{1}{4}(\pi(\Omega) - \pi(H)^2 - 2\pi(H))u_\nu = \frac{1}{4}(\mu - \nu^2 - 2\nu)u_\nu.$$

If $k < 0$, letting $u_\nu = v_{k+1}$ with $\nu = \lambda + 2k + 2$, we get

$$\begin{aligned}\pi(E)v_k &= \pi(E)\pi(F)v_{k+1} \\ &= \frac{1}{4}(\mu - (\lambda + 2k + 2)^2 + 2(\lambda + 2k + 2))v_{k+1}.\end{aligned}$$

If $k > 0$, letting $u_\nu = v_{k-1}$ with $\nu = \lambda + 2k - 2$, we get

$$\begin{aligned}\pi(F)v_k &= \pi(F)\pi(E)v_{k-1} \\ &= \frac{1}{4}(\mu - (\lambda + 2k - 2)^2 - 2(\lambda + 2k - 2))v_{k-1}.\end{aligned}$$

□

Corollary 24. *Let V be an irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module. Then the following conditions are equivalent:*

1. *There exists a $\lambda \in \mathbb{C}$ such that $\dim_{\mathbb{C}} \ker(\pi(H) - \lambda) = 1$;*
2. *There exist a non-zero $v_0 \in V$ that is simultaneously an eigenvector for $\pi(H)$ and $\pi(\Omega)$, i.e.*

$$\pi(H)v_0 = \lambda v_0 \quad \text{and} \quad \pi(\Omega)v = \mu v \quad \text{for some } \lambda, \mu \in \mathbb{C}.$$

(Some authors start with one condition and some with the other. Now we know that they are equivalent.) In the context of representation theory, it is customary to call the eigenvalues of $\pi(H)$ *weights* of V and the eigenspaces of $\pi(H)$ *weight spaces* of V . We summarize the results of this subsection as follows:

Proposition 25. *Let V be an irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module with such that one of the two equivalent conditions from Corollary 24 is satisfied. Then V is a direct sum of weight*

spaces, all weight spaces of V are one-dimensional, and the weights of V are of the form $\lambda + 2k$ with k ranging over an “interval of integers”

$$\mathbb{Z} \cap [a, b], \quad \mathbb{Z} \cap [a, \infty), \quad \mathbb{Z} \cap (-\infty, b] \quad \text{or} \quad \mathbb{Z}.$$

Moreover, there exists a $\mu \in \mathbb{C}$ such that $\pi(\Omega)v = \mu v$ for all $v \in V$.

4.2 Classification of Irreducible $\mathfrak{sl}(2, \mathbb{C})$ -Modules

First, we consider the case when the irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module V is finite-dimensional.

Proposition 26. *Let V be an irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module of dimension $d + 1$. Then V has a basis $\{v_0, v_1, \dots, v_d\}$ such that*

$$\begin{aligned} \pi(H)v_k &= (-d + 2k)v_k, & 0 \leq k \leq d, \\ \pi(E)v_k &= v_{k+1}, & 0 \leq k < d, & \quad \pi(E)v_d = 0, \\ \pi(F)v_k &= k(d + 1 - k)v_{k-1}, & 0 < k \leq d, & \quad \pi(F)v_0 = 0. \end{aligned}$$

Moreover,

$$\pi(\Omega)v = d(d + 2)v, \quad \forall v \in V,$$

and V is determined up to isomorphism by its dimension.

We denote the irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module of dimension $d + 1$ by F_d .

Proof. The operator $\pi(\Omega)$ commutes with all $\pi(X)$, $X \in \mathfrak{sl}(2, \mathbb{C})$. Hence, by Schur’s Lemma, there exists a $\mu \in \mathbb{C}$ such that $\pi(\Omega)v = \mu v$ for all $v \in V$. (This is one of the places where we use the finite-dimensionality of V .) Since V is finite-dimensional,

$\pi(H)$ has at least one eigenvalue. Let $\lambda \in \mathbb{C}$ be an eigenvalue for $\pi(H)$ with the least real part and $v_0 \in V$ a corresponding eigenvector. By Lemma 19, $\pi(F)v_0 = 0$. By Lemma 23, the set of nonzero $v_k = (\pi(E))^k v_0$, $k = 0, 1, 2, 3, \dots$, form a basis for V . Since $\dim V = d + 1$, $\{v_0, \dots, v_d\}$ is a basis such that

$$\pi(E)v_k = v_{k+1}, \quad 0 \leq k < d, \quad \pi(E)v_d = 0$$

and

$$\pi(H)v_k = (\lambda + 2k)v_k, \quad 0 \leq k \leq d.$$

Since $\pi(F)v_0 = 0$,

$$\begin{aligned} \mu v_0 &= \pi(\Omega)v_0 = (\pi(H)^2 - 2\pi(H) + 4\pi(E)\pi(F))v_0 \\ &= (\lambda^2 - 2\lambda)v_0 = \lambda(\lambda - 2)v_0 \end{aligned}$$

and $\mu = \lambda(\lambda - 2)$. Similarly, from $\pi(E)v_d = 0$ we obtain

$$\begin{aligned} \mu v_d &= \pi(\Omega)v_d = (\pi(H)^2 + 2\pi(H) + 4\pi(F)\pi(E))v_d \\ &= ((\lambda + 2d)^2 + 2(\lambda + 2d))v_d = (\lambda + 2d)(\lambda + 2d + 2)v_d \end{aligned}$$

and $\mu = (\lambda + 2d)(\lambda + 2d + 2)$. Solving

$$\lambda(\lambda - 2) = (\lambda + 2d)(\lambda + 2d + 2)$$

we obtain

$$\lambda = -d \quad \text{and} \quad \mu = \lambda(\lambda - 2) = d(d + 2).$$

The property

$$\pi(F)v_k = k(d + 1 - k)v_{k-1}, \quad 0 < k \leq d,$$

follows from Lemma 23:

$$\begin{aligned}
& \frac{1}{4}(\mu - (\lambda + 2k - 2)^2 - 2(\lambda + 2k - 2)) \\
&= \frac{1}{4}(d(d + 2) - (-d + 2k - 2)^2 - 2(-d + 2k - 2)) \\
&= \frac{1}{4}(d^2 + 2d - d^2 - (2k - 2)^2 + 4dk - 4d + 2d - 2(2k - 2)) \\
&= -(k - 1)^2 + dk - k + 1 = k(d + 1 - k).
\end{aligned}$$

This finishes our proof of the theorem. \square

Remark 27. *It is also true that any indecomposable finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -module is irreducible. Hence every finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -module is a direct sum of irreducible submodules. See the book [9] for details.*

Example 28. *The irreducible finite-dimensional representations of $SL(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$ can be constructed as follows. Let $SL(2, \mathbb{C})$ act on \mathbb{C}^2 by matrix multiplication:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}), \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2.$$

Let

$$V = \{\text{polynomial functions on } \mathbb{C}^2\},$$

$$V_d = \{\text{homogeneous polynomials on } \mathbb{C}^2 \text{ of degree } d\},$$

and define a representation π of $SL(2, \mathbb{C})$ in V by

$$(\pi(g)f)(z) = f(g^{-1} \cdot z), \quad g \in SL(2, \mathbb{C}), \quad f \in V, \quad z \in \mathbb{C}^2.$$

It is easy to see that the subspaces V_d remain invariant under this action. Thus

$$(\pi, V) = \bigoplus_{d \geq 0} (\pi_d, V_d),$$

where π_d denotes the restriction of π to V_d . Differentiating, one obtains representations (π_d, V_d) of $\mathfrak{sl}(2, \mathbb{C})$. Then each $\mathfrak{sl}(2, \mathbb{C})$ -module V_d has dimension $d + 1$, is irreducible, and hence isomorphic to F_d (homework).

Remark 29. One can extend this construction to any closed subgroup of $G \subset GL(n, \mathbb{C})$. However, the resulting representations are not necessarily irreducible. If G is a complex analytic subgroup of $GL(n, \mathbb{C})$ and the Lie algebra of G is simple, each irreducible representation of G appears as a subrepresentation of the space of homogeneous polynomials on \mathbb{C}^n of degree d , for some d .

Now we look at the irreducible infinite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -modules. By Proposition 25, the weights of V are of the form $\lambda + 2k$ with k ranging over an infinite “interval of integers”

$$\mathbb{Z} \cap [a, \infty), \quad \mathbb{Z} \cap (-\infty, b] \quad \text{or} \quad \mathbb{Z}.$$

In the first case we call V a *lowest weight module* and in the second – a *highest weight module*. The irreducible modules of the last type do not have a universally accepted name, but one could call them *irreducible principal series modules*. First we classify the irreducible lowest weight modules.

Proposition 30. *Let V be an irreducible infinite-dimensional lowest weight $\mathfrak{sl}(2, \mathbb{C})$ -module. Then V has a basis of H -eigenvectors $\{v_0, v_1, v_2, \dots\}$ and a $\lambda \in \mathbb{C}$, $\lambda \neq 0, -1, -2, \dots$, such that*

$$\begin{aligned} \pi(H)v_k &= (\lambda + 2k)v_k, & 0 \leq k, \\ \pi(E)v_k &= v_{k+1}, & 0 \leq k, \\ \pi(F)v_k &= -k(\lambda + k - 1)v_{k-1}, & 0 < k, \quad \pi(F)v_0 = 0. \end{aligned}$$

Moreover,

$$\pi(\Omega)v = \lambda(\lambda - 2)v, \quad \forall v \in V,$$

and V is determined up to isomorphism by its lowest weight λ .

We denote the irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module of lowest weight λ by V_λ .

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue for $\pi(H)$ with the least real part and $v_0 \in V$ a corresponding eigenvector. By Lemma 19, $\pi(F)v_0 = 0$. By Lemma 23, the set of nonzero $v_k = (\pi(E))^k v_0$, $k = 0, 1, 2, 3, \dots$, form a basis for V . By assumption, V is infinite-dimensional, hence we get a basis $\{v_0, v_1, v_2, \dots\}$ such that

$$\pi(E)v_k = v_{k+1}, \quad 0 \leq k, \quad \pi(F)v_0 = 0$$

and

$$\pi(H)v_k = (\lambda + 2k)v_k, \quad 0 \leq k.$$

Since $\pi(F)v_0 = 0$,

$$\begin{aligned} \pi(\Omega)v_0 &= (\pi(H)^2 - 2\pi(H) + 4\pi(E)\pi(F))v_0 \\ &= (\lambda^2 - 2\lambda)v_0 = \lambda(\lambda - 2)v_0 \end{aligned}$$

and it follows that $\pi(\Omega)v = \lambda(\lambda - 2)v$, for all $v \in V$. The property

$$\pi(F)v_k = -k(\lambda + k - 1)v_{k-1}, \quad 0 < k,$$

follows from Lemma 23:

$$\begin{aligned}
& \frac{1}{4}(\mu - (\lambda + 2k - 2)^2 - 2(\lambda + 2k - 2)) \\
&= \frac{1}{4}(\lambda(\lambda - 2) - (\lambda + 2k - 2)^2 - 2(\lambda + 2k - 2)) \\
&= \frac{1}{4}(\lambda^2 - 2\lambda - \lambda^2 - (2k - 2)^2 - 4k\lambda + 4\lambda - 2\lambda - 2(2k - 2)) \\
&= -(k - 1)^2 - k\lambda - k + 1 = -k(\lambda + k - 1).
\end{aligned}$$

It remains to show $\lambda \neq 0, -1, -2, \dots$. If λ is an integer and $\lambda \leq 0$, consider $k = -\lambda + 1$, then

$$\pi(F)v_k = -k(\lambda + k - 1)v_{k-1} = 0,$$

so $\{v_{-\lambda+1}, v_{-\lambda+2}, v_{-\lambda+3}, \dots\}$ form a basis for a proper $\mathfrak{sl}(2, \mathbb{C})$ -submodule of V (isomorphic to $V_{-\lambda+2}$). This contradicts V being irreducible. \square

The classification of irreducible highest weight modules is similar.

Proposition 31. *Let V be an irreducible infinite-dimensional highest weight $\mathfrak{sl}(2, \mathbb{C})$ -module. Then V has a basis of H -eigenvectors $\{\bar{v}_0, \bar{v}_1, \bar{v}_2, \dots\}$ and a $\lambda \in \mathbb{C}$, $\lambda \neq 0, 1, 2, \dots$, such that*

$$\begin{aligned}
\pi(H)\bar{v}_k &= (\lambda - 2k)\bar{v}_k, & 0 \leq k, \\
\pi(F)\bar{v}_k &= \bar{v}_{k+1}, & 0 \leq k, \\
\pi(E)\bar{v}_k &= k(\lambda - k + 1)\bar{v}_{k-1}, & 0 < k, \quad \pi(E)\bar{v}_0 = 0.
\end{aligned}$$

Moreover,

$$\pi(\Omega)v = \lambda(\lambda + 2)v, \quad \forall v \in V,$$

and V is determined up to isomorphism by its highest weight λ .

We denote the irreducible infinite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -module of highest weight λ by \bar{V}_λ . Finally, we turn our attention to what we call the irreducible principal series modules.

Proposition 32. *Let V be an irreducible infinite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -module that satisfies one of the two equivalent conditions from Corollary 24 and is not a highest nor lowest weight module. Let $v_0 \in V$ be an eigenvector for H with eigenvalue λ , then there is a $\mu \in \mathbb{C}$ such that*

$$\pi(\Omega)v = \mu v, \quad \forall v \in V,$$

and V has a basis of eigenvectors $\{\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots\}$ such that

$$\begin{aligned} \pi(H)v_k &= (\lambda + 2k)v_k, & k \in \mathbb{Z}, \\ \pi(E)v_k &= v_{k+1} \quad \text{if } k \geq 0, & \pi(F)v_k = v_{k-1} \quad \text{if } k \leq 0, \\ \pi(E)v_k &= \frac{1}{4}(\mu - (\lambda + 2k + 1)^2 + 1)v_{k+1} \quad \text{if } k < 0, \\ \pi(F)v_k &= \frac{1}{4}(\mu - (\lambda + 2k - 1)^2 + 1)v_{k-1} \quad \text{if } k > 0. \end{aligned}$$

The constants $\lambda, \mu \in \mathbb{C}$ are subject to the constraint

$$\lambda \pm \sqrt{\mu + 1} \neq \text{odd integer}. \quad (1)$$

We denote such an irreducible infinite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -module by $P(\lambda, \mu)$. Two $\mathfrak{sl}(2, \mathbb{C})$ -modules $P(\lambda, \mu)$ and $P(\lambda', \mu')$ are isomorphic if and only if $\mu' = \mu$ and $\lambda' = \lambda + 2k$, for some $k \in \mathbb{Z}$.

Proof. The statements about the basis and $\mathfrak{sl}(2, \mathbb{C})$ -action follow from Lemma 23. It is easy to see that V is irreducible if and only if

$$\pi(E)v_k \neq 0 \quad \forall k \in \mathbb{Z} \quad \text{and} \quad \pi(F)v_k \neq 0 \quad \forall k \in \mathbb{Z},$$

hence

$$\mu + 1 \neq (\lambda + \text{odd integer})^2$$

and the constraint (1) follows.

If two $\mathfrak{sl}(2, \mathbb{C})$ -modules $P(\lambda, \mu)$ and $P(\lambda', \mu')$ are isomorphic, then $\mu' = \mu$ and $\lambda' = \lambda + 2k$, which follows by examinations of the weights. Conversely, if $\mu' = \mu$ and $\lambda' = \lambda + 2k$, the module $P(\lambda', \mu')$ has λ as a weight, so let $v_0 \in P(\lambda', \mu')$ be an eigenvector for H with eigenvalue λ , then Lemma 23 implies $P(\lambda, \mu) \simeq P(\lambda', \mu')$. \square

We summarize the results of this section as follows:

Theorem 33. *There are precisely four types of irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module such that one of the two equivalent conditions from Corollary 24 is satisfied:*

- *Finite-dimensional ones F_d of dimension $d+1$ with weights*

$$\begin{array}{cccccc} [\circ & \circ & \dots & \circ & \circ] \\ -d & -d+2 & \dots & d-2 & d \end{array}$$

$$\pi(\Omega)v = d(d+2)v \text{ for all } v \in F_d.$$

- *Lowest weight modules V_λ of lowest weight $\lambda \in \mathbb{C}$, $\lambda \neq 0, -1, -2, \dots$, with weights*

$$\begin{array}{cccc} [\circ & \circ & \circ & \dots \\ \lambda & \lambda+2 & \lambda+4 & \dots \end{array}$$

$$\pi(\Omega)v = \lambda(\lambda-2)v \text{ for all } v \in V_\lambda.$$

- *Highest weight modules \bar{V}_λ of highest weight $\lambda \in \mathbb{C}$, $\lambda \neq 0, 1, 2, \dots$, with weights*

$$\begin{array}{cccc} \dots & \circ & \circ & \circ] \\ \dots & \lambda-4 & \lambda-2 & \lambda \end{array}$$

$$\pi(\Omega)v = \lambda(\lambda + 2)v \text{ for all } v \in \bar{V}_\lambda.$$

- Irreducible principal series modules $P(\lambda, \mu)$ with $\lambda, \mu \in \mathbb{C}$ subject to the constraint

$$\lambda \pm \sqrt{\mu + 1} \neq \text{odd integer}$$

and weights

$$\begin{array}{ccccccc} \dots & & \circ & & \circ & & \circ & & \dots \\ \dots & & \lambda - 2 & & \lambda & & \lambda + 2 & & \dots \end{array}$$

$$\pi(\Omega)v = \mu v \text{ for all } v \in P(\lambda, \mu).$$

Proof. We already know that any irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module has to be of the type F_d , V_λ , \bar{V}_λ or $P(\lambda, \mu)$. It remains to prove the existence part. I.e., for example, that for any choice of the parameters $\lambda, \mu \in \mathbb{C}$ such that

$$\lambda \pm \sqrt{\mu + 1} \neq \text{odd integer},$$

there *exists* an irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module of type $P(\lambda, \mu)$. To do that, let

$$V = \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \cdot v_k$$

and *define* the action of $\mathfrak{sl}(2, \mathbb{C})$ by

$$\pi(H)v_k = (\lambda + 2k)v_k, \quad k \in \mathbb{Z},$$

$$\pi(E)v_k = v_{k+1} \quad \text{if } k \geq 0, \quad \pi(F)v_k = v_{k-1} \quad \text{if } k \leq 0,$$

$$\pi(E)v_k = \frac{1}{4}(\mu - (\lambda + 2k + 1)^2 + 1)v_{k+1} \quad \text{if } k < 0,$$

$$\pi(F)v_k = \frac{1}{4}(\mu - (\lambda + 2k - 1)^2 + 1)v_{k-1} \quad \text{if } k > 0.$$

Then check

$$\begin{aligned} [\pi(H), \pi(E)] &= 2\pi(E), & [\pi(H), \pi(F)] &= -2\pi(F), \\ [\pi(E), \pi(F)] &= \pi(H), \end{aligned}$$

which proves that the action of $\mathfrak{sl}(2, \mathbb{C})$ on V is well-defined. Finally, the constraint (1) ensures this $\mathfrak{sl}(2, \mathbb{C})$ -module V is irreducible. \square

Remark 34. *There are many infinite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -modules that are indecomposable, but not irreducible. The book [9] describes them in detail. We will see examples of those when we discuss the principal series of $SL(2, \mathbb{R})$.*

Of these irreducible $\mathfrak{sl}(2, \mathbb{C})$ -modules, only the finite-dimensional ones “lift” to representations of $SL(2, \mathbb{C})$. Precisely those modules with $\lambda \in \mathbb{Z}$ occur as underlying $\mathfrak{sl}(2, \mathbb{C})$ -modules of irreducible representations of $SU(1, 1)$.

5 The Complexification, the Cartan Decomposition and Maximal Compact Subgroups

In this section we state without proofs the properties of maximal compact subgroups. For more details and proofs see the books [13] and [8].

5.1 Assumptions on Groups

Definition 35. *A Lie algebra \mathfrak{g} is simple if it has no proper ideals and $\dim \mathfrak{g} > 1$. A Lie algebra \mathfrak{g} is semisimple if it can be written as a direct sum of simple ideals \mathfrak{g}_i ,*

$$\mathfrak{g} = \bigoplus_{1 \leq i \leq N} \mathfrak{g}_i.$$

One calls a Lie algebra \mathfrak{g} reductive if it can be written as a direct sum of ideals

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z},$$

with \mathfrak{s} semisimple and \mathfrak{z} = center of \mathfrak{g} . A Lie group is simple, respectively semisimple, if it has finitely many connected components and if its Lie algebra is simple, respectively semisimple. A closed linear group is a closed Lie subgroup $G \subset GL(n, \mathbb{R})$ or $G \subset GL(n, \mathbb{C})$.

In this course we always assume that our group is a *linear connected semisimple real Lie group* and denote such a group by $G_{\mathbb{R}}$. Thus $G_{\mathbb{R}}$ denotes a closed connected Lie subgroup of some $GL(n, \mathbb{R})$ with semisimple Lie algebra. We skip the definition of a reductive Lie group – it is technical and requires more than just the Lie algebra being reductive. Most results that are true for linear connected semisimple real Lie groups also extend to linear semisimple real Lie groups that are not necessarily connected as well as linear reductive real Lie groups, but the proofs may become more involved.

Example 36. *The Lie groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$ are simple; the Lie groups $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ are reductive. Any compact real Lie group is a closed linear group, as can be deduced from the Peter-Weyl Theorem (requires some effort), and is moreover reductive. Any compact real Lie group with discrete center is closed linear semisimple. On the other hand, the universal covering $\widetilde{SL}(n, \mathbb{R})$ of $SL(n, \mathbb{R})$, $n \geq 2$, is not a linear group.*

We conclude this subsection with a list of equivalent characterizations of semisimple Lie algebras.

Proposition 37. *The following conditions on a Lie algebra \mathfrak{g} are equivalent to \mathfrak{g} being semisimple:*

(a) \mathfrak{g} can be written as a direct sum of simple ideals \mathfrak{g}_i ,

$$\mathfrak{g} = \bigoplus_{1 \leq i \leq N} \mathfrak{g}_i;$$

(b) \mathfrak{g} contains no nonzero solvable ideals;

(c) The radical of \mathfrak{g} , $\text{rad}(\mathfrak{g})$, is zero;

(d) The Killing form of \mathfrak{g}

$$K(X, Y) = \text{Tr}(\text{ad } X, \text{ad } Y), \quad X, Y \in \mathfrak{g},$$

is non-degenerate.

5.2 Complexifications and Real Forms

Let $G_{\mathbb{R}}$ be a closed linear connected semisimple real Lie group. Like any linear Lie group, $G_{\mathbb{R}}$ has a *complexification* – a complex Lie group G , with Lie algebra

$$\mathfrak{g} =_{\text{def}} \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}},$$

containing $G_{\mathbb{R}}$ as a Lie subgroup, connected and such that the inclusion $G_{\mathbb{R}} \hookrightarrow G$ induces $\mathfrak{g}_{\mathbb{R}} \hookrightarrow \mathfrak{g}$, $X \mapsto 1 \otimes X$.

To construct a complexification, one regards $G_{\mathbb{R}}$ as subgroup of $GL(n, \mathbb{R})$, so that $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{gl}(n, \mathbb{R})$. That makes \mathfrak{g} a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{gl}(n, \mathbb{R})$. Then let G be the connected Lie subgroup of $GL(n, \mathbb{C})$ with Lie algebra \mathfrak{g} . Since $G_{\mathbb{R}}$ is connected, G contains $G_{\mathbb{R}}$. When G is a complexification of $G_{\mathbb{R}}$, one calls $G_{\mathbb{R}}$ a *real form* of G .

In general, the complexification of a linear Lie group depends on the realization of that group as a linear group. In our situation, $G_{\mathbb{R}}$ is closed linear connected semisimple and the complexification G is determined by $G_{\mathbb{R}}$ up to isomorphism, but the *embedding* $G \subset GL(n, \mathbb{C})$ still depends on the realization $G_{\mathbb{R}} \subset GL(n, \mathbb{R})$ of $G_{\mathbb{R}}$ as real group.

Example 38. *Let $G_{\mathbb{R}}$ be $SL(2, \mathbb{R}) \subset GL(2, \mathbb{R})$ or $SU(1, 1) \subset GL(2, \mathbb{C}) \subset GL(4, \mathbb{R})$, then its complexification is isomorphic to $SL(2, \mathbb{C})$.*

Theorem 39. *Let G be a connected complex semisimple Lie group. Then G has a compact real form $U_{\mathbb{R}}$, i.e. a compact connected real subgroup $U_{\mathbb{R}} \subset G$ such that G is a complexification of $U_{\mathbb{R}}$. Moreover, any two compact real forms $U_{\mathbb{R}}$ and $U'_{\mathbb{R}}$ are conjugate to each other by an element of G , i.e. there exists an element $g \in G$ such that $U'_{\mathbb{R}} = g \cdot U_{\mathbb{R}} \cdot g^{-1}$.*

Example 40. *A compact real form of $SL(n, \mathbb{C})$ is $SU(n)$.*

5.3 The Cartan Decomposition

Suppose that $\mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_{\mathbb{R}}$, define a complex conjugation σ of \mathfrak{g} with respect to $\mathfrak{g}_{\mathbb{R}}$ by

$$\sigma(X + iY) = X - iY, \quad X, Y \in \mathfrak{g}_{\mathbb{R}}.$$

Then

$$\sigma(\lambda Z) = \bar{\lambda} \sigma(Z), \quad \forall \lambda \in \mathbb{C}, Z \in \mathfrak{g},$$

(homework). For example, let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(n, \mathbb{R})$, then

$$\sigma \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \bar{a}_{11} & \dots & \bar{a}_{1n} \\ \vdots & \ddots & \vdots \\ \bar{a}_{n1} & \dots & \bar{a}_{nn} \end{pmatrix}$$

is entry-by-entry conjugation. On the other hand, if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(n) \subset \mathfrak{sl}(n, \mathbb{C})$,

$$\sigma(X) = -X^*, \quad \forall X \in \mathfrak{sl}(n, \mathbb{C}),$$

(negative conjugate transpose). In particular, the conjugation σ depends on the choice of a real form $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$.

We start with a closed linear connected semisimple real Lie group $G_{\mathbb{R}}$, take its complexification G and select a compact real form $U_{\mathbb{R}} \subset G$:

$$G_{\mathbb{R}} \subset G \supset U_{\mathbb{R}}.$$

Let $\sigma, \tau : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the complex conjugations with respect to $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{u}_{\mathbb{R}}$.

Proposition 41. *By replacing the compact real form $U_{\mathbb{R}} \subset G$ by an appropriate conjugate, one can arrange that $\tau\sigma = \sigma\tau$.*

From now on we fix a $U_{\mathbb{R}} \subset G$ such that $\tau\sigma = \sigma\tau$. Define

$$\theta : \mathfrak{g} \rightarrow \mathfrak{g}, \quad \theta = \tau\sigma = \sigma\tau,$$

this is a complex linear Lie algebra automorphism of \mathfrak{g} (homework). Since $\theta^2 = \tau\sigma\tau\sigma = \tau^2\sigma^2 = Id_{\mathfrak{g}}$, θ is an involution, called *Cartan involution*.

Since θ is an involution, its eigenvalues are ± 1 . Let

$$\mathfrak{k} = +1 \text{ eigenspace of } \theta : \mathfrak{g} \rightarrow \mathfrak{g},$$

$$\mathfrak{p} = -1 \text{ eigenspace of } \theta : \mathfrak{g} \rightarrow \mathfrak{g}.$$

Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

as complex vector spaces. Now let

$$\mathfrak{k}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{k}, \quad \mathfrak{p}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{p},$$

then we have a direct sum decomposition of real vector spaces

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$$

called *Cartan decomposition* of $\mathfrak{g}_{\mathbb{R}}$ (note that we would not have such a decomposition if τ and σ did not commute).

Lemma 42. 1. $\sigma = \tau$ on \mathfrak{k} and $\sigma = -\tau$ on \mathfrak{p} ;

2. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$;

3. $\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$ (direct sum of real vector spaces).

Proof. 1. This part follows from $\theta = \tau\sigma$ and $\theta = +Id$ on \mathfrak{k} , $\theta = -Id$ on \mathfrak{p} .

2. This part follows from the fact that the Cartan involution θ is a Lie algebra isomorphism: $[\theta(X), \theta(Y)] = \theta([X, Y])$ for all $X, Y \in \mathfrak{g}$. In particular, if X belongs to the ε_1 -eigenspace of θ and Y belongs to the ε_2 -eigenspace, where $\varepsilon_1, \varepsilon_2 = \pm 1$, then $[X, Y]$ belongs to the $(\varepsilon_1\varepsilon_2)$ -eigenspace.

3. From the definition of τ ,

$$\mathfrak{u}_{\mathbb{R}} = \{X \in \mathfrak{g}; \tau(X) = X\}.$$

It follows from part 1 that $\mathfrak{k}_{\mathbb{R}}$ and $i\mathfrak{p}_{\mathbb{R}}$ are subspaces of $\mathfrak{u}_{\mathbb{R}}$. Finally, $\mathfrak{k}_{\mathbb{R}} \cap i\mathfrak{p}_{\mathbb{R}} = \{0\}$ and $\dim(\mathfrak{k}_{\mathbb{R}}) + \dim(i\mathfrak{p}_{\mathbb{R}}) = \dim(\mathfrak{g}_{\mathbb{R}}) = \dim(\mathfrak{u}_{\mathbb{R}})$ imply $\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$.

□

Corollary 43. $\mathfrak{k}_{\mathbb{R}}$ is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$ is a $\mathfrak{k}_{\mathbb{R}}$ -module under the ad action. However, $\mathfrak{p}_{\mathbb{R}}$ is not a Lie algebra, unless $[\mathfrak{p}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] = 0$.

Example 44. *Returning to our previous examples, let us take $G_{\mathbb{R}} = SL(n, \mathbb{R})$, its complexification $G = SL(2, \mathbb{C})$ and a compact real form $U_{\mathbb{R}} = SU(n)$. Then the conjugations*

$$\sigma, \tau : \mathfrak{sl}(n, \mathbb{C}) \rightarrow \mathfrak{sl}(n, \mathbb{C}) \quad \text{are} \quad \sigma(X) = \bar{X}, \quad \tau(X) = -X^*,$$

σ and τ commute, so the Cartan involution θ is $\theta(X) = -X^T$ (negative transpose).

$$\mathfrak{k} = \mathfrak{so}(n, \mathbb{C}), \quad \mathfrak{p} = \{\text{symmetric matrices in } \mathfrak{sl}(n, \mathbb{C})\},$$

$$\mathfrak{k}_{\mathbb{R}} = \mathfrak{so}(n, \mathbb{R}), \quad \mathfrak{p}_{\mathbb{R}} = \{\text{symmetric matrices in } \mathfrak{sl}(n, \mathbb{R})\},$$

$\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$ amounts to

$$\mathfrak{su}(n) = \mathfrak{so}(n, \mathbb{R}) \oplus i\{\text{symmetric matrices in } \mathfrak{sl}(n, \mathbb{R})\}.$$

Note that every $X \in \mathfrak{k}_{\mathbb{R}} \subset \mathfrak{u}_{\mathbb{R}}$ is diagonalizable, with purely imaginary eigenvalues. Indeed, $\{t \mapsto \exp tX; t \in \mathbb{R}\}$ is a one-parameter subgroup of $U_{\mathbb{R}}$, and must therefore have bounded matrix entries; that is possible only when X is diagonalizable over \mathbb{C} , with purely imaginary eigenvalues. By the same reasoning, since $i\mathfrak{p}_{\mathbb{R}} \subset \mathfrak{u}_{\mathbb{R}}$, every $X \in \mathfrak{p}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is diagonalizable, with real eigenvalues.

Let $K_{\mathbb{R}} = (U_{\mathbb{R}} \cap G_{\mathbb{R}})^0$ – the connected component of the identity element of $U_{\mathbb{R}} \cap G_{\mathbb{R}}$. Clearly, $K_{\mathbb{R}}$ is a compact connected subgroup of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. The following result describes what is called the *global Cartan decomposition* of $G_{\mathbb{R}}$.

Proposition 45. *The map*

$$K_{\mathbb{R}} \times \mathfrak{p}_{\mathbb{R}} \rightarrow G_{\mathbb{R}}, \quad (k, X) \mapsto k \cdot \exp X,$$

is a diffeomorphism of C^{∞} -manifolds.

Corollary 46. $K_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$ is a strong deformation retract. In particular, $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$ are homotopic to each other, and this inclusion induces isomorphisms of homology and homotopy groups.

Example 47. $SL(2, \mathbb{R}) \simeq SU(1, 1)$ is diffeomorphic to $S^1 \times \mathbb{R}^2$ and homotopic to S^1 .

This diffeomorphism allows us to define the *global Cartan involution* (also denoted by θ)

$$\theta : G_{\mathbb{R}} \rightarrow G_{\mathbb{R}} \quad \text{by} \quad \theta : (k, X) \mapsto (k, -X) \text{ on } K_{\mathbb{R}} \times \mathfrak{p}_{\mathbb{R}}.$$

This global Cartan involution satisfies $\theta^2 = Id_{G_{\mathbb{R}}}$ and its differential at the identity is the Cartan involution $\theta : \mathfrak{g}_{\mathbb{R}} \rightarrow \mathfrak{g}_{\mathbb{R}}$ (which is a Lie algebra automorphism). It follows that the global Cartan involution is a Lie group automorphism. The group $K_{\mathbb{R}}$ can be characterized as the fixed point set of θ , i.e., $K_{\mathbb{R}} = \{g \in G_{\mathbb{R}}; \theta g = g\}$.

Example 48. Returning to the setting of Example 44, we have $K_{\mathbb{R}} = SO(n)$, the Cartan involutions are

$$\theta : \mathfrak{sl}(n, \mathbb{R}) \rightarrow \mathfrak{sl}(n, \mathbb{R}), \quad \theta(X) = -X^T,$$

and

$$\theta : SL(n, \mathbb{R}) \rightarrow SL(n, \mathbb{R}), \quad \theta(g) = (g^{-1})^T.$$

In the setting of this example, the proposition is essentially equivalent to the assertion that any invertible real square matrix can be expressed uniquely as the product of an orthogonal matrix and a positive definite symmetric matrix.

5.4 Maximal Compact Subgroups

Recall that $K_{\mathbb{R}} = (U_{\mathbb{R}} \cap G_{\mathbb{R}})^0$.

Proposition 49. *Let $G_{\mathbb{R}}$ be a closed linear connected semisimple real Lie group, then*

- (a) *Any compact subgroup of $G_{\mathbb{R}}$ is conjugate to a subgroup of $K_{\mathbb{R}}$ by an element of $G_{\mathbb{R}}$;*
- (b) *$K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$;*
- (c) *Any two maximal compact subgroups of $G_{\mathbb{R}}$ are conjugate by an element of $G_{\mathbb{R}}$;*
- (c) *The normalizer of $K_{\mathbb{R}}$ in $G_{\mathbb{R}}$ coincides with $K_{\mathbb{R}}$.*

Note that the properties of maximal compact subgroups of real semisimple closed linear Lie groups are very similar to those of maximal tori of compact groups. Since the maximal compact subgroups are all conjugate, the choice of any one of them is non-essential. At various points, we shall choose a maximal compact subgroup; the particular choice will not matter.

Corollary 50. (a) *All maximal subgroups of $G_{\mathbb{R}}$ are connected;*

- (b) *$K_{\mathbb{R}}$ is the only subgroup of $G_{\mathbb{R}}$, connected or not, that has Lie algebra $\mathfrak{k}_{\mathbb{R}}$;*
- (c) *$K_{\mathbb{R}}$ contains the center of $G_{\mathbb{R}}$.*

Define:

$$K = \text{the complexification of } K_{\mathbb{R}},$$

K is a connected complex Lie subgroup of G – the complexification of $G_{\mathbb{R}}$ – with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. Observe that K cannot

be compact unless $K_{\mathbb{R}} = \{e\}$, which does not happen unless $G_{\mathbb{R}} = \{e\}$. Indeed, any non-zero $X \in \mathfrak{k}_{\mathbb{R}}$ is diagonalizable over \mathbb{C} , with purely imaginary eigenvalues, not all zero, so the complex one-parameter subgroup $\{z \mapsto \exp(zX); z \in \mathbb{C}\}$ of K is unbounded. By construction, the Lie algebras $\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}, \mathfrak{g}, \mathfrak{k}$ and the corresponding groups satisfy the following containments:

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{R}} & \subset & \mathfrak{g} \\ \cup & & \cup \\ \mathfrak{k}_{\mathbb{R}} & \subset & \mathfrak{k} \end{array} \qquad \begin{array}{ccc} G_{\mathbb{R}} & \subset & G \\ \cup & & \cup \\ K_{\mathbb{R}} & \subset & K \end{array}$$

In general, we have a bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of connected compact} \\ \text{semisimple Lie groups } U_{\mathbb{R}} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of connected complex} \\ \text{semisimple Lie groups } G \end{array} \right\}$$

$$U_{\mathbb{R}} \mapsto \text{its complexification } G,$$

$$G \mapsto \text{its compact real form } U_{\mathbb{R}}$$

(in other words, $U_{\mathbb{R}}$ is a maximal compact subgroup of G). The statement remains true if one drops “connected” and replaces “semisimple” with “reductive”.

Since $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}_{\mathbb{R}}$, these two Lie algebras have the same representations over \mathbb{C} – representations can be restricted from \mathfrak{g} to $\mathfrak{u}_{\mathbb{R}}$, and in the opposite direction, can be extended complex linearly. On the global level, these operations induce a canonical bijection

$$\left\{ \begin{array}{l} \text{finite-dimensional} \\ \text{continuous complex} \\ \text{representations of } U_{\mathbb{R}} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{finite-dimensional} \\ \text{holomorphic} \\ \text{representations of } G \end{array} \right\}$$

From a representation of G one gets a representation of $U_{\mathbb{R}}$ by restriction. In the other direction, one uses the fact that continuous finite-dimensional representations of Lie groups are necessarily real analytic, and are determined by the corresponding infinitesimal representations of the Lie algebra. Hence, starting from a finite-dimensional representation of $U_{\mathbb{R}}$, one obtains a representation π of Lie algebras $\mathfrak{u}_{\mathbb{R}}$ and \mathfrak{g} . This representation then lifts to a representation $\tilde{\pi}$ of \tilde{G} – the universal cover of G , which contains $\tilde{U}_{\mathbb{R}}$ – the universal cover of $U_{\mathbb{R}}$ (here we used $U_{\mathbb{R}} \hookrightarrow G$ induces an isomorphism of the fundamental group). Since we started with a representation of $U_{\mathbb{R}}$, the kernel of $\tilde{U}_{\mathbb{R}} \twoheadrightarrow U_{\mathbb{R}}$ acts trivially, hence $\tilde{\pi}$ descends to $\tilde{G}/\ker(\tilde{U}_{\mathbb{R}} \twoheadrightarrow U_{\mathbb{R}}) = G$.

In particular, every finite-dimensional complex representation of $K_{\mathbb{R}}$ extends to a holomorphic representation of K .

6 Definition of a Representation

Interesting representations of non-compact groups are typically infinite-dimensional. To apply analytic and geometric methods, it is necessary to have a topology on the representation space and to impose an appropriate continuity condition on the representation in question. In the finite-dimensional case, there is only one “reasonable” topology and continuity hypothesis, but in the infinite dimensional case, choices must be made. One may want to study both complex and real representations. There is really no loss in treating only the complex case, since one can complexify a real representation and regard the original space as an \mathbb{R} -linear subspace of its complexification.

6.1 A Few Words on Topological Vector Spaces

We shall consider representations on *complete locally convex Hausdorff topological vector spaces over \mathbb{C}* . That includes complex Hilbert spaces, of course. Unitary representations are of particular interest, and one might think that Hilbert spaces constitute a large enough universe of representation spaces. It turns out, however, that even the study of unitary representations naturally leads to the consideration of other types of topological spaces. So, before we give the definition of a representation, we review topological vector spaces.

First of all, all vector spaces are required to be over \mathbb{C} and *complete*, which means every Cauchy sequence converges. (There is a way to define Cauchy sequences for topological vector spaces without reference to any metric.)

Hilbert Spaces: These are complete complex vector spaces with positive-definite inner product, their topology is determined by the metric induced by the inner product.

Example: L^2 -spaces $L^2(X, \mu)$, where (X, μ) is a measure space, the inner product is defined by

$$\langle f, g \rangle = \int f \bar{g} d\mu.$$

Banach Spaces: These are complete complex vector spaces with a norm, their topology is determined by the metric induced by the norm.

Example: L^p -spaces $L^p(X, \mu)$, $1 \leq p \leq \infty$, where (X, μ) is a measure space, the norm is defined by

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p},$$

$$\|f\|_\infty = \inf\{a \geq 0; \mu(\{x \in X; f(x) > a\}) = 0\}, \quad \inf \emptyset = \infty.$$

(We exclude $p < 1$ because the triangle inequality may fail.)

Complete Metric Spaces: These are complete complex vector spaces with a metric, their topology is determined by the metric.

Example: L^p -spaces $L^p(X, \mu)$, $0 < p < 1$, where (X, μ) is a measure space, the metric is defined by

$$d(f, g) = \int |f - g|^p d\mu.$$

Fréchet Spaces: These are complex vector spaces with a countable family of seminorms. Since they occur so often in representation theory, let us review these spaces in more detail.

Let V be a vector space. A *seminorm* on V is a function $p : V \rightarrow [0, \infty)$ such that

- (triangle inequality) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$;
- $p(\lambda x) = |\lambda|p(x)$ for all $x \in V$ and $\lambda \in \mathbb{C}$.

If $p(x) = 0$ implies $x = 0$, then p is a *norm*.

Let V be a vector space with a family of seminorms $\{p_\alpha\}_{\alpha \in A}$. We declare the “balls”

$$U_{x\alpha\varepsilon} = \{v \in V; p_\alpha(v - x) < \varepsilon\}, \quad x \in V, \alpha \in A, \varepsilon > 0,$$

to be open sets and equip V with the topology generated by these sets $U_{x\alpha\varepsilon}$. This topology is Hausdorff if and only if for each $x \in V$, $x \neq 0$, there exists $\alpha \in A$ such that $p_\alpha(x) \neq 0$. If V is Hausdorff and A is countable, then V is metrizable with a translation-invariant metric (i.e. $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in V$). Finally, a *Fréchet space* is a complete Hausdorff topological vector space whose topology is defined

by a countable family of seminorms. (Of course, every Fréchet space is metrizable.)

Examples: Let $U \subset \mathbb{R}^n$ be an open set, and let

$$V_0 = \{\text{continuous functions } f : U \rightarrow \mathbb{C}\}.$$

For each compact set $K \subset U$ we can define a seminorm

$$p_K(f) = \sup_{x \in K} |f(x)|.$$

Of course, this family of seminorms is uncountable, but we can find a sequence of compact sets

$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset U \quad \text{such that} \quad \bigcup_{i=1}^{\infty} K_i = U$$

and use seminorms $p_i = p_{K_i}$ to define topology on V_0 . Then V_0 becomes a Fréchet space (one needs to check the completeness).

We modify the example by letting

$$V_k = \{\mathcal{C}^k \text{ functions } f : U \rightarrow \mathbb{C}\}.$$

For each compact set K and multiindex α , $|\alpha| \leq k$, we can define a seminorm

$$p_{K,\alpha}(f) = \sup_{x \in K} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|.$$

Then the countable family of seminorms $p_{K_i,\alpha}$ turns V_k into a Fréchet space (again, completeness requires verification).

Finally, let

$$V_\infty = \{\mathcal{C}^\infty \text{ functions } f : U \rightarrow \mathbb{C}\}.$$

For each compact set K and multiindex α we have a seminorm

$$p_{K,\alpha}(f) = \sup_{x \in K} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|.$$

Then the countable family of seminorms $p_{K_i, \alpha}$ turns V_∞ into a Fréchet space (of course, completeness is not entirely obvious).

As was mentioned before, we study representations on *complete locally convex Hausdorff topological vector spaces over \mathbb{C}* . But for most purposes Fréchet spaces will suffice. A topological vector space is called *locally convex* if there is a base for the topology consisting of convex sets. Hilbert, Banach and Fréchet spaces are automatically locally convex, but not all metric spaces are locally convex. Local convexity is required to define the integral of vector-valued functions, which is a crucial tool in the study of representations of reductive groups.

Finally, we prefer to work with topological spaces that are *separable*, i.e. have a countable dense subset.

6.2 The Definition

In this subsection $G_{\mathbb{R}}$ can be any real Lie group. Recall that V is a complete locally convex Hausdorff topological vector space over \mathbb{C} such as, for example, a Fréchet space. Let $\text{Aut}(V)$ denote the group of continuous, continuously invertible, linear maps from V to itself; we do not yet specify a topology on this group. There are at least four reasonable notions of continuity one could impose on a homomorphism $\pi : G_{\mathbb{R}} \rightarrow \text{Aut}(V)$:

- a) *continuity*: the action map $G_{\mathbb{R}} \times V \rightarrow V$ is continuous, relative to the product topology on $G_{\mathbb{R}} \times V$;
- b) *strong continuity*: for every $v \in V$, $g \mapsto \pi(g)v$ is continuous as map from $G_{\mathbb{R}}$ to V ;

- c) *weak continuity*: for every $v \in V$ and every l in the continuous linear dual space V^* , the complex-valued function $g \mapsto \langle l, \pi(g)v \rangle$ is continuous;
- d) *continuity in the operator norm*, which makes sense only if V is a Banach space; in that case, $\text{Aut}(V)$ can be equipped with the norm topology, and continuity in the operator norm means that $\pi : G_{\mathbb{R}} \rightarrow \text{Aut}(V)$ is a continuous homomorphism of topological groups.

It is easy to see that

$$\text{continuity} \implies \text{strong continuity} \implies \text{weak continuity},$$

and if V is a Banach space,

$$\text{continuity in the operator norm} \implies \text{continuity}.$$

Remark 51. *By the group property,*

- a) *is equivalent to $G_{\mathbb{R}} \times V \rightarrow V$ being continuous at $(e, 0)$;*
- b) *is equivalent to $g \mapsto \pi(g)v$ being continuous at $g = e$, for every $v \in V$.*

Example 52. *The translation action of $(\mathbb{R}, +)$ on $L^p(\mathbb{R})$ is continuous for $1 \leq p < \infty$, but not continuous in the operator norm; for $p = \infty$ the translation action fails to be continuous, strongly continuous, even weakly continuous.*

The translation action of $(\mathbb{R}, +)$ on $\mathcal{C}^0(\mathbb{R})$ is strongly continuous, but not continuous.

The translation action of $(\mathbb{R}, +)$ on $\mathcal{C}^\infty(\mathbb{R})$ is continuous.

This example shows that requiring continuity in the operator norm is too restrictive – most of the representations of interest

involve translation. Thus, from now on, “representation” shall mean a continuous – in the sense of part (a) above – linear action $\pi : G_{\mathbb{R}} \rightarrow \text{Aut}(V)$ on a complete, locally convex Hausdorff space V .

Example 53. Let $H_{\mathbb{R}} \subset G_{\mathbb{R}}$ be a closed subgroup, suppose that the homogeneous space $G_{\mathbb{R}}/H_{\mathbb{R}}$ has a $G_{\mathbb{R}}$ -invariant measure. Then $G_{\mathbb{R}}$ acts on $L^2(G_{\mathbb{R}}/H_{\mathbb{R}})$ continuously and unitarily. More generally, $G_{\mathbb{R}}$ acts continuously on

$$L^p(G_{\mathbb{R}}/H_{\mathbb{R}}), \quad 1 \leq p < \infty,$$

$$\mathcal{C}^\infty(G_{\mathbb{R}}/H_{\mathbb{R}}), \quad \mathcal{C}_c^\infty(G_{\mathbb{R}}/H_{\mathbb{R}}), \quad \mathcal{C}^{-\infty}(G_{\mathbb{R}}/H_{\mathbb{R}}),$$

where $\mathcal{C}^{-\infty}$ denotes the space of distributions. Later these will be declared infinitesimally equivalent representations.

Remark 54. If $G_{\mathbb{R}}$ is unimodular, then $G_{\mathbb{R}}/H_{\mathbb{R}}$ has a $G_{\mathbb{R}}$ -invariant measure.

A Lie group is called unimodular if it has a bi-invariant Haar measure. The following groups are always unimodular: abelian, compact, connected semisimple, nilpotent.

Proposition 55. Let V be a Banach space, $\pi : G_{\mathbb{R}} \rightarrow \text{Aut}(V)$ a group homomorphism with no continuity hypothesis, then the following are equivalent:

$$\text{continuity} \iff \text{strong continuity} \iff \text{weak continuity}.$$

In this chain of implications, *strong continuity* \implies *continuity* follows relatively easily from the uniform boundedness principle¹, but the implication *weak continuity* \implies *strong continuity*

¹ **Theorem** (Uniform Boundedness Principle). Let V be a Banach space and W be a normed vector space. Suppose that \mathcal{F} is a collection of continuous linear operators from V to W such that

$$\sup_{T \in \mathcal{F}} \|T(x)\| < \infty, \quad \forall x \in V. \quad \text{Then} \quad \sup_{T \in \mathcal{F}} \|T\| < \infty.$$

is more subtle.

If V is a topological vector space, one can equip its continuous linear dual space V^* with something called the strong dual topology. In the case V is a Banach space, this is the topology on V^* defined by the norm

$$\|l\| =_{\text{def}} \sup_{v \in V, \|v\|=1} |l(v)|, \quad l \in V^*.$$

In general, if π is continuous, the induced dual linear action on V^* need not be continuous. However, when V is a reflexive Banach space (i.e. $V \simeq (V^*)^*$), V and V^* play symmetric roles in the definition of weak continuity; in this case, the dual action is also continuous, so there exists a “dual representation” π^* of $G_{\mathbb{R}}$ on the dual Banach space V^* .

An infinite dimensional representation (π, V) typically has numerous invariant subspaces $W \subset V$, but the induced linear action of $G_{\mathbb{R}}$ on V/W is a purely algebraic object unless V/W is Hausdorff, i.e., unless $W \subset V$ is a closed subspace. For this reason, the existence of a non-closed invariant subspace should not be regarded as an obstacle to irreducibility: (π, V) is *irreducible* if V has no proper *closed* $G_{\mathbb{R}}$ -invariant subspaces.

In the same spirit, a *subrepresentation* of (π, V) is a *closed* $G_{\mathbb{R}}$ -invariant subspace $W \subset V$.

A representation (π, V) has *finite length* if every increasing chain of closed $G_{\mathbb{R}}$ -invariant subspaces breaks off after finitely many steps. Equivalently, (π, V) has finite length if there is a finite chain of closed invariant subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_N = V$$

such that each V_i/V_{i-1} is irreducible.

6.3 Admissible Representations

As usual, we assume that $G_{\mathbb{R}}$ is a connected closed linear real Lie group with semisimple Lie algebra. Recall that $K_{\mathbb{R}}$ denotes a maximal compact subgroup of $G_{\mathbb{R}}$.

One calls a representation (π, V) *admissible* if

$$\dim_{\mathbb{R}} \operatorname{Hom}_{K_{\mathbb{R}}}(U, V) < \infty$$

for every finite-dimensional irreducible representation (τ, U) of $K_{\mathbb{R}}$. Informally speaking, admissibility means that the restriction of (π, V) to $K_{\mathbb{R}}$ contains any irreducible $K_{\mathbb{R}}$ -representation only finitely often.

Theorem 56 (Harish-Chandra). *Every irreducible unitary representation (π, V) of $G_{\mathbb{R}}$ is admissible.*

Heuristically, admissible representations of finite length constitute the smallest class that is invariant under “standard constructions” (in a very broad sense!) and contains the irreducible unitary representations. One should regard inadmissible irreducible representations as exotic. All irreducible representations which have come up naturally in geometry, differential equations, physics, and number theory are admissible.

7 Harish-Chandra Modules

The goal of this section is, starting with a representation (π, V) of $G_{\mathbb{R}}$, to construct a representation of $\mathfrak{g}_{\mathbb{R}}$. In the example where $G_{\mathbb{R}} = \mathbb{R}$ acts by translations on $L^p(\mathbb{R})$, $1 \leq p < \infty$, we get an action of the Lie algebra on $f \in L^p(\mathbb{R})$ if and only if f is differentiable. Thus we cannot expect $\mathfrak{g}_{\mathbb{R}}$ act on all of V , and

we want to find a “good” vector subspace of V on which $\mathfrak{g}_{\mathbb{R}}$ can act and which contains enough information about V .

7.1 $K_{\mathbb{R}}$ -finite and \mathcal{C}^{∞} Vectors

Let $U \subset \mathbb{R}^n$ be an open set. Let V be a topological vector space, and consider a function $f : U \rightarrow V$. We can define partial derivatives (with values in V)

$$\frac{\partial f}{\partial x_i} = \lim_{\varepsilon \rightarrow 0} \frac{f(x_1, \dots, x_i + \varepsilon, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{\varepsilon}$$

if the limit exists. Then we can make sense of partial derivatives of higher order and \mathcal{C}^{∞} functions $f : U \rightarrow V$.

If V is a Banach space, we say that a function $f : U \rightarrow V$ is \mathcal{C}^{ω} (or real analytic) if near every point in U it can be represented by an absolutely convergent V -valued power series. (V is required to be a Banach space so we can use the norm to make sense of absolute convergence.) Since $G_{\mathbb{R}}$ “locally looks like \mathbb{R}^n ”, one can make sense of \mathcal{C}^{∞} and \mathcal{C}^{ω} functions $G_{\mathbb{R}} \rightarrow V$.

Definition 57. Let (π, V) be a representation of $G_{\mathbb{R}}$. A vector $v \in V$ is

- a) $K_{\mathbb{R}}$ -finite if v lies in a finite-dimensional $K_{\mathbb{R}}$ -invariant subspace;
- b) a \mathcal{C}^{∞} vector if $g \mapsto \pi(g)v$ is a \mathcal{C}^{∞} map from $G_{\mathbb{R}}$ to V ;
- c) in the case of a Banach space V only, an analytic vector, if $g \mapsto \pi(g)v$ is a \mathcal{C}^{ω} map;
- d) a weakly analytic vector if, for every $l \in V^*$, the complex-valued function $g \mapsto \langle l, \pi(g)v \rangle$ is real analytic.

All reasonable notions of a real analytic V -valued map agree when V is a Banach space, but not for other locally convex topological vector spaces. That is the reason for defining the notion of an analytic vector only in the Banach case. Surprisingly perhaps, even weakly real analytic functions with values in a Banach space are real analytic (i.e., locally representable by absolutely convergent vector valued power series) – see the appendix in [14] for an efficient argument. In the Banach case, then, the notions of an analytic vector and of a weakly analytic coincide; for other representations, the former is not defined, but the latter still makes sense.

As a matter of self-explanatory notation, we write $V_{f\text{ini}}$, V^∞ and V^ω for the spaces of $K_{\mathbb{R}}$ -finite, smooth and weakly analytic vectors in V . We do not equip these spaces with any topology.

Theorem 58 (Harish-Chandra). *If (π, V) is an admissible representation (which may or may not be of finite length),*

- a) $V_{f\text{ini}}$ is a dense subspace of V ;
- b) Every $v \in V_{f\text{ini}}$ is both a \mathcal{C}^∞ vector and a weakly analytic vector;
- c) In particular, V^∞ and V^ω are dense in V .

The theorem applies in particular to $K_{\mathbb{R}}$, considered as maximal compact subgroup of itself. Finite-dimensional subspaces are automatically closed, so the density of $K_{\mathbb{R}}$ -finite vectors forces any infinite dimensional representation (π, V) of $K_{\mathbb{R}}$ to have proper closed invariant subspaces. In other words,

Corollary 59. *Every irreducible representation of $K_{\mathbb{R}}$ is finite dimensional.*

7.2 Proof of Theorem 58

In this subsection we prove Theorem 58. More precisely, we prove that V_{fini} is dense in V and that $V_{fini} \subset V^\infty$, but we omit the $V_{fini} \subset V^\omega$ part. The proof is important because it illustrates essential techniques of representation theory.

Since the group $G_{\mathbb{R}}$ is semisimple, it is unimodular, i.e. has a bi-invariant Haar measure. So fix such a measure dg . It is not unique, since, for example, one can scale it by a positive scalar, but a particular choice of such measure is not essential.

We start with a representation (π, V) and for the moment make no assumptions on admissibility or finite length. Let $f \in \mathcal{C}_c^0(G_{\mathbb{R}})$ (where $\mathcal{C}_c^0(G_{\mathbb{R}})$ denotes the space of continuous complex-valued functions with compact support), define $\pi(f) \in \text{End}(V)$ by

$$\pi(f)v = \int_{g \in G_{\mathbb{R}}} f(g)\pi(g)v \, dg.$$

Note that the integrand $f(g)\pi(g)v$ is a continuous compactly supported function on $G_{\mathbb{R}}$ with values in V , its integral is defined as the limit of Riemannian sums. Since V is complete and locally convex, the limit of these Riemannian sums exists and the integral is well-defined.

Notation: Let l and r denote the linear action of $G_{\mathbb{R}}$ on the spaces of functions on $G_{\mathbb{R}}$ (such as $L^p(G_{\mathbb{R}})$, $\mathcal{C}^k(G_{\mathbb{R}})$, $\mathcal{C}_c^k(G_{\mathbb{R}})$, $0 \leq k \leq \infty$) induced by left and right multiplication:

$$(l(g)f)(h) = f(g^{-1}h), \quad (r(g)f)(h) = f(hg).$$

Lemma 60. a) *If $f \in \mathcal{C}_c^0(G_{\mathbb{R}})$ and $g \in G_{\mathbb{R}}$, then*

$$\pi(g) \circ \pi(f) = \pi(l(g)f), \quad \pi(f) \circ \pi(g) = \pi(r(g^{-1})f);$$

b) For $f_1, f_2 \in \mathcal{C}_c^0(G_{\mathbb{R}})$ we have

$$\pi(f_1) \circ \pi(f_2) = \pi(l(f_1)f_2) = \pi(f_1 * f_2),$$

where $f_1 * f_2$ denotes the convolution product:

$$\begin{aligned} (f_1 * f_2)(h) &= \int_{g \in G_{\mathbb{R}}} f_1(g) f_2(g^{-1}h) dg \\ &= \int_{g \in G_{\mathbb{R}}} f_1(hg^{-1}) f_2(g) dg; \end{aligned}$$

c) Let $\{f_n\} \subset \mathcal{C}_c^0(G_{\mathbb{R}})$ be an approximate identity (this means $f_n \geq 0$, $\int_{G_{\mathbb{R}}} f_n(g) dg = 1$ for all n and $\{\text{supp}(f_n)\} \searrow \{e\}$), then $\pi(f_n) \rightarrow Id_V$ strongly, i.e. $\pi(f_n)v \rightarrow v$ for all $v \in V$.

The proof of this lemma will be left as a homework.

Lemma 61 (Gårding). V^∞ is dense in V .

Proof. Let $\{f_n\} \subset \mathcal{C}_c^\infty(G_{\mathbb{R}})$ be an approximate identity. Then $\pi(f_n)v \rightarrow v$ for all $v \in V$. Thus it suffices to show that $\pi(f)v \in V^\infty$ if $f \in \mathcal{C}_c^\infty(G_{\mathbb{R}})$, i.e. that the map

$$G_{\mathbb{R}} \rightarrow V, \quad g \mapsto \pi(g) \circ \pi(f)v,$$

is smooth. By previous lemma, $\pi(g) \circ \pi(f)v = \pi(l(g)f)v$, and we can break this map as a composition

$$G_{\mathbb{R}} \rightarrow \mathcal{C}_c^\infty(G_{\mathbb{R}}) \rightarrow V, \quad g \mapsto l(g)f \quad \text{and} \quad \tilde{f} \mapsto \pi(\tilde{f})v.$$

For a fixed $v \in V$, the second map

$$\mathcal{C}_c^\infty(G_{\mathbb{R}}) \rightarrow V, \quad \tilde{f} \mapsto \pi(\tilde{f})v,$$

is linear and continuous. Thus it remains to show that, for each $f \in \mathcal{C}_c^\infty(G_{\mathbb{R}})$, the map

$$G_{\mathbb{R}} \rightarrow \mathcal{C}_c^\infty(G_{\mathbb{R}}), \quad g \mapsto l(g)f,$$

is smooth. But this is true essentially because f is smooth. \square

Fix a maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$. Let $\hat{K}_{\mathbb{R}}$ denote the set of isomorphism classes of finite-dimensional irreducible representations of $K_{\mathbb{R}}$ (each of these representations is automatically unitary). Thus, for each $i \in \hat{K}_{\mathbb{R}}$, we have a representation (τ_i, U_i) of $K_{\mathbb{R}}$. Let

$V(i) =_{\text{def}}$ the linear span of all images of $T \in \text{Hom}_{K_{\mathbb{R}}}(U_i, V)$, and call $V(i)$ the *i-isotypic subspace of V*. Then the space of $K_{\mathbb{R}}$ -finite vectors

$$V_{\text{fini}} = \bigoplus_{i \in \hat{K}_{\mathbb{R}}} V(i) \quad (\text{algebraic direct sum}).$$

Let χ_i be the character of (τ_i, U_i) , so

$$\chi_i \in \mathcal{C}^{\infty}(K_{\mathbb{R}}), \quad \chi_i(k) = \text{Tr}(\tau_i(k)), \quad k \in K_{\mathbb{R}},$$

and introduce notations

$$\varphi_i = \dim U_i \cdot \overline{\chi_i}, \quad \pi_{K_{\mathbb{R}}} = \pi|_{K_{\mathbb{R}}}, \quad l_{K_{\mathbb{R}}} = l|_{K_{\mathbb{R}}}, \quad r_{K_{\mathbb{R}}} = r|_{K_{\mathbb{R}}}.$$

We consider $\pi_{K_{\mathbb{R}}}(\varphi_i) \in \text{End}(V)$,

$$\begin{aligned} \pi_{K_{\mathbb{R}}}(\varphi_i)v &=_{\text{def}} \int_{K_{\mathbb{R}}} \varphi_i(k) \pi_{K_{\mathbb{R}}}(k)v \, dk \\ &= \dim U_i \cdot \int_{K_{\mathbb{R}}} \overline{\chi_i}(k) \pi_{K_{\mathbb{R}}}(k)v \, dk, \end{aligned}$$

where dk denotes the bi-invariant Haar measure on $K_{\mathbb{R}}$ normalized by the requirement $\int_{K_{\mathbb{R}}} 1 \, dk = 1$.

We recall Schur Orthogonality Relations:

Theorem 62. Let $(\tau_1, U_1), (\tau_2, U_2)$ be irreducible finite-dimensional unitary representations of a compact Lie group $K_{\mathbb{R}}$ with $K_{\mathbb{R}}$ -invariant inner products $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$. If $x_1, x_2 \in U_1$ and $y_1, y_2 \in U_2$,

$$\begin{aligned} & \int_{K_{\mathbb{R}}} (\tau_1(k)x_1, x_2)_1 \cdot \overline{(\tau_2(k)y_1, y_2)_2} dk \\ &= \begin{cases} 0 & \text{if } (\tau_1, U_1) \not\cong (\tau_2, U_2); \\ \frac{1}{\dim U_1} (x_1, y_1)_1 \cdot \overline{(x_2, y_2)_1} & \text{if } (\tau_1, U_1) = (\tau_2, U_2). \end{cases} \end{aligned}$$

Corollary 63. a) $\tau_i(\varphi_i) = Id_{U_i}$;

b)

$$\chi_i * \chi_j = \begin{cases} 0 & \text{if } i \neq j; \\ \frac{\chi_i}{\dim U_i} & \text{if } i = j, \end{cases} \quad \varphi_i * \varphi_j = \begin{cases} 0 & \text{if } i \neq j; \\ \varphi_i & \text{if } i = j. \end{cases}$$

Proof. a) Homework.

b) Let $\{x_m\}$ and $\{y_n\}$ be orthonormal bases for U_i and U_j relative to the $K_{\mathbb{R}}$ -invariant inner products $(\cdot, \cdot)_i$ and $(\cdot, \cdot)_j$. Write

$$(\chi_i * \chi_j)(h) = \int_{K_{\mathbb{R}}} \chi_i(k) \chi_j(k^{-1}h) dk.$$

Then

$$\begin{aligned} \chi_i(k) \chi_j(k^{-1}h) &= \sum_{m,n} (\tau_i(k)x_m, x_m)_i \cdot (\tau_j(k^{-1}h)y_n, y_n)_j \\ &= \sum_{m,n,p} (\tau_i(k)x_m, x_m)_i \cdot (\tau_j(k^{-1})y_p, y_n)_j \cdot (\tau_j(h)y_n, y_p)_j \\ &= \sum_{m,n,p} (\tau_i(k)x_m, x_m)_i \cdot \overline{(\tau_j(k)y_n, y_p)_j} \cdot (\tau_j(h)y_n, y_p)_j, \end{aligned}$$

so Schur Orthogonality Theorem implies

$$\int_{K_{\mathbb{R}}} (\tau_i(k)x_m, x_m)_i \cdot \overline{(\tau_j(k)y_n, y_p)_j} dk \quad \text{and} \quad \chi_i * \chi_j$$

are 0 when $i \neq j$. When $i = j$, we can take the basis $\{y_n\}$ to be the same as $\{x_m\}$, then, by Schur orthogonality,

$$\begin{aligned} & \int_{K_{\mathbb{R}}} \chi_i(k) \chi_j(k^{-1}h) dk \\ &= \frac{1}{\dim U_i} \sum_{m,n,p} (x_m, x_n)_i \cdot \overline{(x_m, x_p)_i} \cdot (\tau_j(h)x_n, x_p)_i \\ &= \frac{1}{\dim U_i} \sum_p (\tau_j(h)x_p, x_p)_i = \frac{\chi_i(h)}{\dim U_i}. \end{aligned}$$

□

Lemma 64. a)

$$\pi_{K_{\mathbb{R}}}(\varphi_i) \circ \pi_{K_{\mathbb{R}}}(\varphi_j) = \begin{cases} 0 & \text{if } i \neq j; \\ \pi_{K_{\mathbb{R}}}(\varphi_i) & \text{if } i = j; \end{cases}$$

- b) *The image of $\pi_{K_{\mathbb{R}}}(\varphi_i)$ is $V(i)$ and $\pi_{K_{\mathbb{R}}}(\varphi_i)$ restricted to $V(i)$ is the identity map. In other words, $\pi_{K_{\mathbb{R}}}(\varphi_i)$ is a projection onto $V(i)$;*
- c) *V_{fini} is dense in V (we make no assumption on admissibility of V).*

Proof. a) We have:

$$\pi_{K_{\mathbb{R}}}(\varphi_i) \circ \pi_{K_{\mathbb{R}}}(\varphi_j) = \pi_{K_{\mathbb{R}}}(\varphi_i * \varphi_j) = \begin{cases} 0 & \text{if } i \neq j; \\ \pi_{K_{\mathbb{R}}}(\varphi_i) & \text{if } i = j. \end{cases}$$

b) Recall that

$V(i)$ = the linear span of all images of $T \in \text{Hom}_{K_{\mathbb{R}}}(U_i, V)$.

Then, for all $x \in U_i$,

$$\begin{aligned} (\pi_{K_{\mathbb{R}}}(\varphi_i) \circ T)x &= \int_{K_{\mathbb{R}}} \varphi_i(k) (\pi_{K_{\mathbb{R}}}(k) \circ T)x \, dk \\ &= T \circ \int_{K_{\mathbb{R}}} \varphi_i(k) \tau_i(k)x \, dk = (T \circ \tau_i(\varphi_i))x = Tx, \end{aligned}$$

since $\tau_i(\varphi_i) = Id_{U_i}$. This proves that the image of $\pi_{K_{\mathbb{R}}}(\varphi_i)$ contains $V(i)$ and that $\pi_{K_{\mathbb{R}}}(\varphi_i)$ restricted to $V(i)$ is the identity map.

On the other hand, the collection of functions $\{l_{K_{\mathbb{R}}}(k)\varphi_i\}$, $k \in K_{\mathbb{R}}$, is a linear combination of matrix coefficients

$$\overline{(\tau_i(k^{-1}h)x_m, x_n)_i} = (\tau_i(k)x_n, \tau_i(h)x_m)_i,$$

where $\{x_m\}$ is a basis for U_i and $(\cdot, \cdot)_i$ is a $K_{\mathbb{R}}$ -invariant inner product, hence spans a finite-dimensional vector subspace in $\mathcal{C}^{\infty}(K_{\mathbb{R}})$. In particular, for all $v \in V$, the collection of vectors

$$\pi_{K_{\mathbb{R}}}(k)(\pi_{K_{\mathbb{R}}}(\varphi_i)v) = \pi_{K_{\mathbb{R}}}(l_{K_{\mathbb{R}}}(k)\varphi_i)v, \quad k \in K_{\mathbb{R}},$$

spans a finite-dimensional subspace of V . Hence

$$\pi_{K_{\mathbb{R}}}(\varphi_i)v \in V_{fini} = \bigoplus_{j \in \hat{K}_{\mathbb{R}}} V(j).$$

By part (a),

$$\pi_{K_{\mathbb{R}}}(\varphi_i)|_{V(j)} = 0 \quad \text{if } i \neq j.$$

Since $\pi_{K_{\mathbb{R}}}(\varphi_i) \circ \pi_{K_{\mathbb{R}}}(\varphi_i) = \pi_{K_{\mathbb{R}}}(\varphi_i)$, it follows that $\text{Im } \pi_{K_{\mathbb{R}}}(\varphi_i) \subset V(i)$.

- c) Recall the Stone-Weierstrass Theorem: Let X be a compact Hausdorff space and let $\mathcal{C}^0(X)$ denote the space of continuous complex-valued functions on X . Suppose A is a subalgebra of $\mathcal{C}^0(X)$ such that it contains a non-zero constant function, closed under conjugation and separates points. Then A is dense in $\mathcal{C}^0(X)$.

By Stone-Weierstrass Theorem, $l(K_{\mathbb{R}})$ -finite functions are dense in $\mathcal{C}^0(K_{\mathbb{R}})$. Indeed, the $l(K_{\mathbb{R}})$ -finite functions include matrix coefficients of finite-dimensional representations of $K_{\mathbb{R}}$. The fact that they separate points follows from Peter-Weyl Theorem.

Using the approximate identity argument, one can show that the set

$$\{\pi_{K_{\mathbb{R}}}(f)v; f \in \mathcal{C}^0(K_{\mathbb{R}}), v \in V\}$$

is dense in V . Therefore, the set

$$\{\pi_{K_{\mathbb{R}}}(f)v; f \in \mathcal{C}^0(K_{\mathbb{R}}), f \text{ is } l(K_{\mathbb{R}})\text{-finite}, v \in V\} \quad (2)$$

is also dense in V . But any vector in the set (2) is automatically in V_{fini} .

□

Corollary 65. *For each $i \in \hat{K}_{\mathbb{R}}$, $V^{\infty} \cap V(i)$ is dense in $V(i)$.*

Proof. Let $v \in V(i)$. Choose an approximate identity $\{f_n\} \subset \mathcal{C}_c^{\infty}(G_{\mathbb{R}})$. Then $\pi(f_n) \rightarrow v$, hence

$$\pi_{K_{\mathbb{R}}}(\varphi_i) \circ \pi(f_n)v \rightarrow \pi_{K_{\mathbb{R}}}(\varphi_i)v = v.$$

But

$$\pi_{K_{\mathbb{R}}}(\varphi_i) \circ \pi(f_n) = \pi(l_{K_{\mathbb{R}}}(\varphi_i)f_n) \text{ and } l_{K_{\mathbb{R}}}(\varphi_i)f_n \in \mathcal{C}_c^{\infty}(G_{\mathbb{R}}).$$

Hence each $\pi_{K_{\mathbb{R}}}(\varphi_i) \circ \pi(f_n)v \in V^{\infty} \cap V(i)$.

□

Recall that a representation (π, V) is admissible if $\dim V(i) < \infty$ for all $i \in \hat{K}_{\mathbb{R}}$.

Corollary 66. *If (π, V) is admissible, $V_{fini} \subset V^\infty$.*

Recall that a topological vector space is separable if it has a countable dense subset. If (π, V) is admissible, combining vector space bases for each $V(i)$, $i \in \hat{K}_{\mathbb{R}}$, results in a countable set of linearly independent vectors in V such that their linear combinations are dense in V . Thus, if (π, V) is admissible, V is automatically separable.

Finally, we comment that, in order to prove that all $K_{\mathbb{R}}$ -finite vectors are weakly analytic, one can show that the functions $g \mapsto \langle l, \pi(g)v \rangle$, for $v \in V_{fini}$ and $l \in V^*$, satisfy elliptic differential equations with C^ω coefficients, which implies they are real analytic. To construct such an operator, use the Casimir elements $\Omega_G \in \mathcal{U}(\mathfrak{g})$ (its symbol is hyperbolic) and $\Omega_K \in \mathcal{U}(\mathfrak{k}) \subset \mathcal{U}(\mathfrak{g})$ (its symbol is semidefinite, but degenerate) and show that $\Omega_G - 2\Omega_K \in \mathcal{U}(\mathfrak{g})$ is elliptic. Then argue that some polynomial of $\Omega_G - 2\Omega_K$ annihilates $\langle l, \pi(g)v \rangle$.

7.3 Harish-Chandra Modules

For $v \in V^\infty$ and $X \in \mathfrak{g}_{\mathbb{R}}$, define

$$\pi(X)v =_{\text{def}} \left. \frac{d}{dt} \pi(\exp(tX))v \right|_{t=0},$$

then $\pi(X) : V^\infty \rightarrow V^\infty$. This defines a representation of $\mathfrak{g}_{\mathbb{R}}$ on V^∞ . Complexifying, we obtain a representation of \mathfrak{g} on V^∞ . The action of \mathfrak{g} extends to the universal enveloping algebra, thus V^∞ is a $\mathcal{U}(\mathfrak{g})$ -module.

Lemma 67. *Whether or not (π, V) is admissible, $V^\infty \cap V_{fini}$ is \mathfrak{g} -invariant.*

Proof. Let $\mathfrak{g} \otimes V^\infty \rightarrow V^\infty$ be the action map

$$X \otimes v \mapsto \left. \frac{d}{dt} \pi(\exp(tX))v \right|_{t=0},$$

This map is $K_{\mathbb{R}}$ -equivariant, i.e., for all $k \in K_{\mathbb{R}}$,

$$\begin{aligned} \text{Ad}(k)X \otimes \pi(k)v &\mapsto \left. \frac{d}{dt} \pi(\exp(t \text{Ad}(k)X))\pi(k)v \right|_{t=0} \\ &= \left. \frac{d}{dt} \pi(k) \circ \pi(\exp(tX))v \right|_{t=0} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\pi(k) \circ \pi(\exp(\varepsilon X))v - \pi(k)v}{\varepsilon} \\ &= \pi(k) \lim_{\varepsilon \rightarrow 0} \frac{\pi(\exp(\varepsilon X))v - v}{\varepsilon} = \pi(k) \left. \frac{d}{dt} \pi(\exp(tX))v \right|_{t=0}. \end{aligned}$$

Hence, if $v \in V^\infty$ is $K_{\mathbb{R}}$ -finite (i.e. lies in a finite-dimensional $K_{\mathbb{R}}$ -invariant subspace), then so is $\pi(X)v$. \square

Corollary 68. *If (π, V) is admissible, V_{fini} is a $\mathcal{U}(\mathfrak{g})$ -module.*

From now on we assume that the representation (π, V) is admissible. Not only $\mathcal{U}(\mathfrak{g})$ acts on V_{fini} , but also $K_{\mathbb{R}}$. As a $K_{\mathbb{R}}$ -representation,

$$V_{fini} = \bigoplus_{i \in \hat{K}_{\mathbb{R}}} V(i), \quad \text{each } V(i) \text{ is finite-dimensional.}$$

Recall that K is a complexification of $K_{\mathbb{R}}$ and the restriction from K to $K_{\mathbb{R}}$ gives a bijection

$$\left\{ \begin{array}{c} \text{finite-dimensional} \\ \text{holomorphic} \\ \text{representations of } K \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{finite-dimensional} \\ \text{continuous complex} \\ \text{representations of } K_{\mathbb{R}} \end{array} \right\}$$

We conclude that K also acts on V_{fini} .

Caution: K does *not* act on V !

Even though V_{fini} has no natural Hausdorff topology – it is not closed in V unless $\dim V < \infty$ – it makes sense to say that K acts holomorphically on V_{fini} : like $K_{\mathbb{R}}$, K acts *locally finitely*, in the sense that every vector lies in a finite dimensional invariant subspace; the invariant finite dimensional subspaces do carry natural Hausdorff topologies, and K does act holomorphically on them. The Lie algebra \mathfrak{k} has two natural actions on V_{fini} , by differentiation of the K -action, and via the inclusion $\mathfrak{k} \subset \mathfrak{g}$ and the $\mathcal{U}(\mathfrak{g})$ -module structure. These two actions coincide, essentially by construction. Moreover, for all $X \in \mathcal{U}(\mathfrak{g})$, $v \in V_{fini}$ and $k \in K$,

$$\pi(k)(Xv) = (\text{Ad}(k)X)(\pi(k)v), \quad (3)$$

as can be deduced from the well known formula $\exp(\text{Ad } kX) = k \exp(X) k^{-1}$, for $X \in \mathfrak{g}_{\mathbb{R}}$, $k \in K_{\mathbb{R}}$.

Definition 69. A (\mathfrak{g}, K) -module is a complex vector space M , equipped with the structure of $\mathcal{U}(\mathfrak{g})$ -module and with a linear action of K such that:

- a) The action of K is locally finite, i.e., every $m \in M$ lies in a finite-dimensional K -invariant subspace on which K acts holomorphically;
- b) When the K -action is differentiated, the resulting action of \mathfrak{k} agrees with the action of \mathfrak{k} on M via $\mathfrak{k} \hookrightarrow \mathfrak{g}$ and the $\mathcal{U}(\mathfrak{g})$ -module structure.
- c) The identity (3) holds for all $k \in K$, $X \in \mathcal{U}(\mathfrak{g})$, $v \in M$.

Definition 70. A Harish-Chandra module is a (\mathfrak{g}, K) -module M which is finitely generated over $\mathcal{U}(\mathfrak{g})$ and admissible, in the sense that every irreducible K -representation occurs in M with finite multiplicity.

Remark 71. If K is connected (which is the case if $G_{\mathbb{R}}$ is connected), the compatibility condition (c) follows from condition (b). Indeed, it is sufficient to verify (3) on the infinitesimal level, in which case it becomes, by (b),

$$ZXv = [Z, X]v + XZv, \quad \forall X \in \mathcal{U}(\mathfrak{g}), Z \in \mathfrak{k},$$

which is true.

But in the case of non-connected group $G_{\mathbb{R}}$, each maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ meets every connected component of $G_{\mathbb{R}}$, hence also disconnected. In this case the compatibility condition (b) does not imply (3), and the compatibility condition (c) must be stated separately.

The discussion leading up to the definition shows that the space of $K_{\mathbb{R}}$ -finite vectors V_{fini} of an admissible representation (π, V) is an admissible (\mathfrak{g}, K) -module. In order to show that V_{fini} is a Harish-Chandra module, we need to show that V_{fini} is finitely-generated over $\mathcal{U}(\mathfrak{g})$. This part requires that V has finite length and a few more steps. Recall a part of Theorem 58:

Theorem 72. $V_{fini} \subset V^{\omega}$. In other words, for each $v \in V_{fini}$ and $l \in V^*$, the matrix coefficient function $G_{\mathbb{R}} \rightarrow \mathbb{C}$, $g \mapsto \langle l, \pi(g)v \rangle$, is real-analytic.

Corollary 73. Let (π, V) be an admissible representation of $G_{\mathbb{R}}$ and let W be a submodule of the (\mathfrak{g}, K) -module V_{fini} . Then \overline{W} – the closure of W in V – is $G_{\mathbb{R}}$ -invariant (hence a subrepresentation).

Proof. Suppose $w \in W$, we need to show that $\pi(g)w$ lies in \overline{W} , for all $g \in G_{\mathbb{R}}$. Equivalently, we must show that the matrix coefficient $g \mapsto \langle l, \pi(g)v \rangle$ is identically zero for all l in the annihilator of W in V^* . By Theorem 72, this function is real-analytic. W being \mathfrak{g} -invariant implies that all derivatives at the identity element e of this function vanish. Hence the function is zero. \square

Lemma 74. *Let (π, V) be an admissible representation of $G_{\mathbb{R}}$, then we have a bijection*

$$\left\{ \begin{array}{c} \text{closed } G_{\mathbb{R}}\text{-invariant} \\ \text{subspaces of } V \end{array} \right\} \simeq \left\{ \begin{array}{c} (\mathfrak{g}, K)\text{-submodules} \\ \text{of } V_{fini} \end{array} \right\}$$

$$\begin{array}{ll} \text{subrepresentation } \tilde{V} \subset V & \mapsto \tilde{V}_{fini} \subset V_{fini} \\ (\mathfrak{g}, K)\text{-submodule } W \subset V_{fini} & \mapsto \text{closure } \overline{W} \subset V. \end{array}$$

Proof. If $\tilde{V} \subset V$ is a closed $G_{\mathbb{R}}$ -invariant subspace, then $\overline{\tilde{V}_{fini}} = \tilde{V}$, since \tilde{V}_{fini} is dense in \tilde{V} .

Conversely, if $W \subset V_{fini}$ is a (\mathfrak{g}, K) -submodule, then $(\overline{W})_{fini} = W$. Indeed, we certainly have $(\overline{W})_{fini} \supset W$. If $(\overline{W})_{fini} \neq W$, there exists an $i \in \hat{K}_{\mathbb{R}}$ such that $\overline{W}(i) \supsetneq W(i)$. Let $w \in \overline{W}(i)$, then there exists a sequence $\{w_n\}$ in W converging to w . Apply $\pi_{K_{\mathbb{R}}}(\varphi_i)$ to get

$$\pi_{K_{\mathbb{R}}}(\varphi_i)w_n \rightarrow \pi_{K_{\mathbb{R}}}(\varphi_i)w = w, \quad \pi_{K_{\mathbb{R}}}(\varphi_i)w_n \in W(i)$$

(here we used that W is K -invariant). Hence

$$\overline{W}(i) \subset \overline{W(i)} = W(i),$$

since $W(i)$ is finite-dimensional. \square

Corollary 75. *A representation (π, V) is irreducible if and only if V_{fini} is irreducible as (\mathfrak{g}, K) -module.*

Corollary 76. *Let (π, V) be an admissible representation of $G_{\mathbb{R}}$ of finite length, then V_{fini} is a Harish-Chandra module.*

Proof. We need to show that V_{fini} is finitely generated as a $\mathcal{U}(\mathfrak{g})$ -module. Observe that if $w_1, \dots, w_n \in V_{fini}$, then the (\mathfrak{g}, K) -submodule W generated by w_1, \dots, w_n is finitely generated as a $\mathcal{U}(\mathfrak{g})$ -module. Indeed, by the compatibility condition (c) of Definition 69, as a $\mathcal{U}(\mathfrak{g})$ -module, W is generated by

$$\{\pi(k_1)w_1, \dots, \pi(k_n)w_n; k_1, \dots, k_n \in K\},$$

and those vectors span a finite-dimensional vector subspace $W_0 \subset W$. Then a basis of W_0 generates W over $\mathcal{U}(\mathfrak{g})$.

If V_{fini} is *not* finitely generated, we get an infinite chain of $\mathcal{U}(\mathfrak{g})$ -modules and hence an infinite chain of (\mathfrak{g}, K) -submodules

$$W_1 \subsetneq W_2 \subsetneq W_3 \subsetneq \dots \subset V_{fini}.$$

Take their closures in V :

$$\overline{W_1} \subset \overline{W_2} \subset \overline{W_3} \subset \dots \subset V.$$

By previous lemma, all inclusions are proper, so we get an infinite chain of closed $G_{\mathbb{R}}$ -invariant subspaces of V . But this contradicts to V being of finite length. \square

From now on, we write $\mathrm{HC}(V)$ for the space of $K_{\mathbb{R}}$ -finite vectors of an admissible representation (π, V) and call $\mathrm{HC}(V)$ *the Harish-Chandra module of π* . The next statement formalizes the properties of Harish-Chandra modules we have mentioned so far:

Theorem 77 (Harish-Chandra). *The association*

$$V \mapsto \mathrm{HC}(V) = V_{\mathrm{fini}}$$

establishes a covariant, exact², faithful³ functor

$$\begin{aligned} & \left\{ \begin{array}{l} \text{category of admissible } G_{\mathbb{R}}\text{-representations} \\ \text{of finite length and } G_{\mathbb{R}}\text{-equivariant maps} \end{array} \right\} \\ & \xrightarrow{\mathrm{HC}} \left\{ \begin{array}{l} \text{category of Harish-Chandra modules} \\ \text{and } (\mathfrak{g}, K)\text{-equivariant linear maps} \end{array} \right\}. \end{aligned}$$

7.4 Infinitesimal Equivalence

Definition 78. *Two admissible representations of finite length (π_1, V_1) and (π_2, V_2) are infinitesimally equivalent or infinitesimally isomorphic if $\mathrm{HC}(V_1) \simeq \mathrm{HC}(V_2)$.*

Loosely speaking, infinitesimal equivalence means that the two representations are the same except for the choice of topology. This is a good notion of equivalence because, for example, if two irreducible unitary representations are infinitesimally equivalent, they are isomorphic as unitary representations.

Example: Let $G_{\mathbb{R}} = SU(1, 1)$. Recall that

$$SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} ; a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\},$$

$G_{\mathbb{R}}$ has $G = SL(2, \mathbb{C})$ as complexification, and is conjugate in G to $SL(2, \mathbb{R})$. As maximal compact subgroup, we choose the diagonal subgroup, in which case its complexification also

²A functor is *exact* if it preserves exact sequences.

³A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is *faithful* if, for all $X, Y \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is injective.

consists of diagonal matrices:

$$K_{\mathbb{R}} = \left\{ k_{\theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}; \theta \in \mathbb{R} \right\} \simeq SO(2) \simeq U(1),$$

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a \in \mathbb{C}^{\times} \right\} \simeq \mathbb{C}^{\times}.$$

The group $SU(1, 1)$ acts transitively by fractional linear transformations on the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}.$$

Since the isotropy subgroup at the origin is $K_{\mathbb{R}}$,

$$\mathbb{D} \simeq SU(1, 1)/K_{\mathbb{R}}.$$

The group $SU(1, 1)$ acts on $\mathcal{C}^{\infty}(\mathbb{D})$ by left translations:

$$(l(g)f)(z) = f(g^{-1} \cdot z), \quad f \in \mathcal{C}^{\infty}(\mathbb{D}), \quad z \in \mathbb{D}, \quad (4)$$

and on the subspace

$$H^2(\mathbb{D}) \stackrel{\text{def}}{=} \begin{array}{l} \text{space of holomorphic functions on } \mathbb{D} \\ \text{with } L^2 \text{ boundary values,} \end{array}$$

topologized by the inclusion $H^2(\mathbb{D}) \hookrightarrow L^2(S^1)$. One can show that both actions are representations, i.e., they are continuous with respect to the natural topologies of the two spaces.

Note that

$$(l(k_{\theta})f)(z) = f(e^{-2i\theta} \cdot z), \quad l(k_{\theta})z^n = e^{-2in\theta} z^n.$$

Hence, $f \in H^2(\mathbb{D})$ is $K_{\mathbb{R}}$ -finite if and only if f has a finite Taylor series at the origin, i.e., if and only if f is a polynomial:

$$H^2(\mathbb{D})_{\text{fini}} = \mathbb{C}[z].$$

In particular, $(l, H^2(D))$ is admissible. This representation is not irreducible, since $H^2(\mathbb{D})$ contains the constant functions \mathbb{C} as an obviously closed invariant subspace. It does have finite length; in fact, the quotient $H^2(\mathbb{D})/\mathbb{C}$ is irreducible, as follows from a simple infinitesimal calculation in the Harish-Chandra module $\mathrm{HC}(H^2(\mathbb{D})) = \mathbb{C}[z]$.

Besides $V = H^2(\mathbb{D})$, the action (4) on each of the following spaces, equipped with the natural topology in each case, defines a representation of $SU(1, 1)$:

- a) $H^p(\mathbb{D})$ = space of holomorphic functions on \mathbb{D} with L^p boundary values, $1 \leq p \leq \infty$;
- b) $H^\infty(\mathbb{D})$ = space of holomorphic functions on \mathbb{D} with \mathcal{C}^∞ boundary values;
- c) $H^{-\infty}(\mathbb{D})$ = space of holomorphic functions on \mathbb{D} with distribution boundary values;
- d) $H^\omega(\mathbb{D})$ = space of holomorphic functions on \mathbb{D} with real analytic boundary values;
- e) $H^{-\omega}(\mathbb{D})$ = space of all holomorphic functions on \mathbb{D} .

Taking boundary values, one obtains inclusions $H^p(\mathbb{D}) \hookrightarrow L^p(S^1)$, which are equivariant with respect to the action of $SU(1, 1)$ on $L^p(S^1)$ by linear fractional transformations. The latter fails to be continuous when $p = \infty$, but that is not the case for the image of $H^\infty(\mathbb{D})$ in $L^\infty(S^1)$. One can show that $H^\infty(\mathbb{D})$ is the space of C^∞ vectors for the Hilbert space representation $(l, H^2(\mathbb{D}))$.

Arguing as in the case of $H^2(\mathbb{D})$, one finds that the representation l of $SU(1, 1)$ on each of the spaces a)-e) has $\mathbb{C}[z]$ as

Harish-Chandra module, so all of them are infinitesimally equivalent. This is the typical situation, not just for $SU(1,1)$, but for all groups $G_{\mathbb{R}}$ of the type we are considering: every infinite-dimensional admissible representation (π, V) of finite length is infinitesimally equivalent to an infinite family of pairwise non-isomorphic representations.

7.5 A Few Words about Globalization

One can ask the following very natural question: “Does every Harish-Chandra module arise as the space of $K_{\mathbb{R}}$ -finite vectors of some admissible representation of $G_{\mathbb{R}}$ of finite length?” In other words: “Does every Harish-Chandra module M have a *globalization*, i.e. an admissible $G_{\mathbb{R}}$ -representation (π, V) of finite length, such that $\mathrm{HC}(V) = M$?” The answer to this question is affirmative and is due to W.Casselman (~ 1980), but the proof is quite complicated and indirect.

Then it is natural to ask if a globalization can be chosen in a functorial manner – in other words, whether the functor HC in Theorem 77 has a right inverse. Such functorial globalizations do exist. Four of them are of particular interest, the C^{∞} and $C^{-\infty}$ globalizations due to W.Casselman and N.Wallach, as well as the minimal globalization and the maximal globalization due to M.Kashiwara and W.Schmid. All four are *topologically exact*, i.e., they map exact sequences of Harish-Chandra modules into exact sequences of representations in which every morphism has *closed range*. The main technical obstacle in constructing the canonical globalizations is to establish this closed range property.

We describe without proofs the maximal globalization functor

due to M.Kashiwara and W.Schmid [10].

Definition 79. *Let M be a Harish-Chandra module. By the maximal globalization we mean a representation $\mathrm{MG}(M)$ of $G_{\mathbb{R}}$ such that $\mathrm{HC}(\mathrm{MG}(M)) = M$ and, for any other globalization V of M , the identity map $M \rightarrow M$ extends to a $G_{\mathbb{R}}$ -equivariant continuous linear map $V \rightarrow \mathrm{MG}(M)$.*

Similarly, one can define the minimal globalization.

We start with a Harish-Chandra module M . Let M^* denote the vector space of all complex linear functions on M . Since M does not have any topology, neither does M^* . Define

$$M' =_{\mathrm{def}} (M^*)_{\mathrm{fini}},$$

then \mathfrak{g} and K act on M' making it a (\mathfrak{g}, K) -module and, in fact, a Harish-Chandra module. We call M' the dual Harish-Chandra module of M .

Define

$$\widetilde{\mathrm{MG}}(M) =_{\mathrm{def}} \mathrm{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', \mathcal{C}^{\infty}(G_{\mathbb{R}})),$$

where the $(\mathfrak{g}, K_{\mathbb{R}})$ -action on $\mathcal{C}^{\infty}(G_{\mathbb{R}})$ is induced by the action of $G_{\mathbb{R}}$ by multiplications on the left:

$$(l(g)f)(h) = f(g^{-1}h).$$

Then we have a linear action of $G_{\mathbb{R}}$ on $\widetilde{\mathrm{MG}}(M)$ induced by the action of $G_{\mathbb{R}}$ on $\mathcal{C}^{\infty}(G_{\mathbb{R}})$ by multiplications on the right:

$$(r(g)f)(h) = f(hg).$$

Moreover, $\widetilde{\mathrm{MG}}(M)$ has a natural Fréchet space topology. Indeed, fix a vector space basis $\{m_i\}$ of M' , it is automatically

countable. Pick a diffeomorphism between an open set U of \mathbb{R}^n and an open set in $G_{\mathbb{R}}$, and let $C \subset U$ be a compact subset. Then smooth functions on $G_{\mathbb{R}}$ can be “restricted” to U and it makes sense to talk about their partial derivatives. Finally, let α be a multiindex. Then the topology of $\widetilde{\text{MG}}(M)$ is induced by seminorms

$$p_{C,\alpha,i}(f) = \sup_{x \in C} \left| \frac{\partial^{|\alpha|} f(m_i)}{\partial x^\alpha}(x) \right|, \quad f \in \widetilde{\text{MG}}(M),$$

with the compact sets C ranging over a countable family covering $G_{\mathbb{R}}$. The linear action of $G_{\mathbb{R}}$ on $\widetilde{\text{MG}}(M)$ is continuous with respect to this topology.

The space $\widetilde{\text{MG}}(M)$ remains unchanged if one replaces $\mathcal{C}^\infty(G_{\mathbb{R}})$ by the space of real-analytic functions $\mathcal{C}^\omega(G_{\mathbb{R}})$:

Lemma 80. *The inclusion $\mathcal{C}^\omega(G_{\mathbb{R}}) \hookrightarrow \mathcal{C}^\infty(G_{\mathbb{R}})$ induces a topological isomorphism*

$$\text{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', \mathcal{C}^\omega(G_{\mathbb{R}})) \simeq \text{Hom}_{(\mathfrak{g}, K_{\mathbb{R}})}(M', \mathcal{C}^\infty(G_{\mathbb{R}})).$$

Theorem 81 (M.Kashiwara and W.Schmid, 1994). *The representation $\widetilde{\text{MG}}(M)$ of $G_{\mathbb{R}}$ is the maximal globalization of the Harish-Chandra module M . The functor $M \mapsto \widetilde{\text{MG}}(M)$ is the right adjoint⁴ of HC and is topologically exact.*

7.6 Unitary Representations and Infinitesimal Equivalence

Theorem 82 (Harish-Chandra). *If two irreducible unitary representations are infinitesimally equivalent⁵, they are iso-*

⁴ Let \mathcal{C}, \mathcal{D} be two categories, and let $F : \mathcal{D} \rightarrow \mathcal{C}$, $G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. Suppose there is a family of bijections

$$\text{Hom}_{\mathcal{C}}(FY, X) \cong \text{Hom}_{\mathcal{D}}(Y, GX)$$

which is natural in the variables X and Y . Then the functor F is called a left adjoint functor, while G is called a right adjoint functor.

⁵By Theorem 56 irreducible unitary representations are automatically admissible, so it makes sense to talk about their underlying Harish-Chandra modules.

morphic as unitary representations.

Proof. Let (π_1, V_1) and (π_2, V_2) be two irreducible unitary infinitesimally equivalent representations with $G_{\mathbb{R}}$ -invariant inner products $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$, and let W be the common Harish-Chandra module. Restricting, we get two (possibly different) inner products on W , also denoted by $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$. For all $X \in \mathfrak{g}_{\mathbb{R}}$, $u, v \in W$ and $\alpha = 1, 2$, we have

$$0 = \frac{d}{dt} (\pi_{\alpha}(\exp tX)u, \pi_{\alpha}(\exp tX)v)_{\alpha} \Big|_{t=0} = (Xu, v)_{\alpha} + (u, Xv)_{\alpha}.$$

Complexifying, we get

$$(Xu, v)_{\alpha} + (u, \overline{X}v)_{\alpha} = 0, \quad X \in \mathfrak{g}, \quad \alpha = 1, 2,$$

where \overline{X} denotes complex conjugate of X relative to $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$.

Also, for both inner products, applying Schur's lemma for $K_{\mathbb{R}}$, we get

$$W(i) \perp W(j), \quad i \neq j.$$

Hence $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ define conjugate-linear isomorphisms

$$W \simeq W^* = \bigoplus_{i \in \hat{K}_{\mathbb{R}}} W(i)^*.$$

Compose one isomorphism with the inverse of the other to get a linear isomorphism $W \simeq W$ which, by construction, relates the two inner products and is (\mathfrak{g}, K) -equivariant.

This implies $V_1 \simeq V_2$ as Hilbert spaces, and this isomorphism commutes with the actions of $K_{\mathbb{R}}$ on V_{α} and \mathfrak{g} on $(V_{\alpha})_{fini}$, $\alpha = 1, 2$. We need to show that this isomorphism commutes with the actions of $G_{\mathbb{R}}$ as well. For this purpose we use Theorem 72, which says that every $K_{\mathbb{R}}$ -finite vector is weakly analytic.

Suppose that this isomorphism $V_1 \simeq V_2$ identifies

$$(V_1)_{fini} \ni u_1 \leftrightarrow u_2 \in (V_2)_{fini}, \quad (V_1)_{fini} \ni v_1 \leftrightarrow v_2 \in (V_2)_{fini}.$$

Then the matrix coefficient functions

$$(\pi_1(g)u_1, v_1)_1 \quad \text{and} \quad (\pi_2(g)u_2, v_2)_2, \quad g \in G_{\mathbb{R}},$$

are real-analytic and have the same derivatives at the identity element, hence coincide and

$$V_1 \ni \pi_1(g)u_1 \leftrightarrow \pi_2(g)u_2 \in V_2. \quad \forall g \in G_{\mathbb{R}}.$$

By continuity, the isomorphism $V_1 \simeq V_2$ commutes with the actions of $G_{\mathbb{R}}$. \square

8 Construction of Representations of $SU(1, 1)$

In this section we construct representations of $SU(1, 1)$ following the book [20]. This construction is very similar to that of irreducible finite-dimensional representations of $SL(2, \mathbb{C})$ given in Example 28.

8.1 Construction of Representations of $SL(2, \mathbb{R})$

The construction requires a parameter

$$\chi = (l, \varepsilon), \quad \text{where } l \in \mathbb{C} \quad \text{and} \quad \varepsilon \in \{0, 1/2\}.$$

The $SL(2, \mathbb{R})$ realization can be constructed as follows. Let $SL(2, \mathbb{R})$ act on $\mathbb{R}^2 \setminus \{0\}$ by matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}.$$

Let \tilde{V}_χ denote the space of \mathbb{C} -valued functions $\varphi(x_1, x_2)$ on $\mathbb{R}^2 \setminus \{0\}$ with the following properties:

- $\varphi(x_1, x_2)$ is smooth;
- $\varphi(ax_1, ax_2) = a^{2l} \cdot \varphi(x_1, x_2)$, for all $a \in \mathbb{R}$, $a > 0$, i.e. φ is homogeneous of homogeneity degree $2l$;
- φ is even if $\varepsilon = 0$ and odd if $\varepsilon = 1/2$:

$$\varphi(-x_1, -x_2) = (-1)^{2\varepsilon} \cdot \varphi(x_1, x_2).$$

Then $SL(2, \mathbb{R})$ acts on \tilde{V}_χ by

$$\begin{aligned} (\tilde{\pi}_\chi(g)\varphi)(x_1, x_2) &= \varphi(g^{-1} \cdot (x_1, x_2)), \\ g &\in SL(2, \mathbb{R}), \quad \varphi \in \tilde{V}_\chi, \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}. \end{aligned}$$

It is easy to see that \tilde{V}_χ remains invariant under this action, so this action is well-defined.

The space \tilde{V}_χ is a closed subspace of the Fréchet space of all smooth functions on $\mathbb{R}^2 \setminus \{0\}$, hence inherits Fréchet space topology. The $SL(2, \mathbb{R})$ -action is continuous with respect to this topology, thus we get a representation $(\tilde{\pi}_\chi, \tilde{V}_\chi)$ of $SL(2, \mathbb{R})$.

8.2 Construction of Representations of $SU(1, 1)$

If we want to do any computations with the representation $(\tilde{\pi}_\chi, \tilde{V}_\chi)$ of $SL(2, \mathbb{R})$, it is much more convenient to rewrite it as a representation of $SU(1, 1)$.

Recall that

$$SU(1, 1) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SL(2, \mathbb{C}); \ a, b \in \mathbb{C}, \ |a|^2 - |b|^2 = 1 \right\}.$$

We need to replace \mathbb{R}^2 with a real 2-dimensional $SU(1,1)$ -invariant subspace of \mathbb{C}^2 . Note that the real subspace

$$\left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \in \mathbb{C}^2; z \in \mathbb{C} \right\}$$

is preserved by $SU(1,1)$:

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} az + b\bar{z} \\ \bar{b}z + \bar{a}\bar{z} \end{pmatrix} = \begin{pmatrix} az + b\bar{z} \\ \overline{az + b\bar{z}} \end{pmatrix}.$$

As before, we use a parameter

$$\chi = (l, \varepsilon), \quad \text{where } l \in \mathbb{C} \quad \text{and} \quad \varepsilon \in \{0, 1/2\}.$$

Let V_χ denote the space of \mathbb{C} -valued functions $\varphi(z)$ on $\mathbb{C} \setminus \{0\}$ with the following properties:

- $\varphi(z)$ is a smooth function of x and y , where $z = x + iy$;
- $\varphi(az) = a^{2l} \cdot \varphi(z)$, for all $a \in \mathbb{R}$, $a > 0$, i.e. φ is homogeneous of homogeneity degree $2l$;
- φ is even if $\varepsilon = 0$ and odd if $\varepsilon = 1/2$:

$$\varphi(-z) = (-1)^{2\varepsilon} \cdot \varphi(z).$$

Then $SU(1,1)$ acts on V_χ by

$$\begin{aligned} (\pi_\chi(g)\varphi)(z) &= \varphi(g^{-1} \cdot z), \\ g &\in SU(1,1), \quad \varphi \in V_\chi, \quad z \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

Explicitly, if $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, then $g^{-1} = \begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix}$, and

$$(\pi_\chi(g)\varphi)(z) = \varphi(g^{-1} \cdot z) = \varphi(\bar{a}z - b\bar{z}), \quad \varphi \in V_\chi, z \in \mathbb{C} \setminus \{0\}.$$

It is easy to see that V_χ remains invariant under this action, so this action is well-defined. As before, V_χ is a Fréchet space with topology inherited from the space of all smooth functions on $\mathbb{C} \setminus \{0\}$, and the $SU(1,1)$ -action is continuous with respect to this topology. Thus we get a representation (π_χ, V_χ) of $SU(1,1)$.

Now, let us turn our attention to the space V_χ . Every homogeneous function on $\mathbb{C} \setminus \{0\}$ is completely determined by its values on the unit circle $S^1 = \{z \in \mathbb{C}; |z| = 1\}$:

$$\varphi(z) = |z|^{2l} \cdot \varphi(z/|z|), \quad z/|z| \in S^1.$$

Conversely, every smooth function on S^1 satisfying $\varphi(-z) = (-1)^{2\varepsilon} \cdot \varphi(z)$, $z \in S^1$, extends to a homogeneous function of degree $2l$ which is an element of V_χ .

It will be convenient to realize the space V_χ in another way on the circle. Namely, for $\varepsilon = 0$ with each function $\varphi(z)$ we associate a function $f(e^{i\theta})$, defined by

$$f(e^{i\theta}) =_{\text{def}} \varphi(e^{i\theta/2}), \quad \theta \in \mathbb{R}.$$

Since $\varphi(z)$ is even, the function $f(e^{i\theta})$ is uniquely defined and smooth. If $\varepsilon = 1/2$ we take

$$f(e^{i\theta}) =_{\text{def}} e^{i\theta/2} \cdot \varphi(e^{i\theta/2}), \quad \theta \in \mathbb{R}.$$

This function is uniquely defined and smooth, since $\varphi(z)$ is odd. In this way, for any $\chi(l, \varepsilon)$ the space V_χ can be realized as the space $\mathcal{C}^\infty(S^1)$.

Let us find the expression for $\pi_\chi(g)$ under this identification of V_χ with $\mathcal{C}^\infty(S^1)$. Assume first that $\varepsilon = 0$. Let $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$,

we have:

$$\begin{aligned} (\pi_\chi(g)f)(e^{i\theta}) &= (\pi_\chi(g)\varphi)(e^{i\theta/2}) = \varphi(\bar{a}e^{i\theta/2} - be^{-i\theta/2}) \\ &= |\bar{a}e^{i\theta/2} - be^{-i\theta/2}|^{2l} \cdot \varphi\left(\frac{\bar{a}e^{i\theta/2} - be^{-i\theta/2}}{|\bar{a}e^{i\theta/2} - be^{-i\theta/2}|}\right). \end{aligned}$$

We denote

$$\frac{\bar{a}e^{i\theta/2} - be^{-i\theta/2}}{|\bar{a}e^{i\theta/2} - be^{-i\theta/2}|} \quad \text{by} \quad e^{i\psi/2}, \quad \text{then} \quad e^{i\psi} = \frac{\bar{a}e^{i\theta} - b}{-\bar{b}e^{i\theta} + a}.$$

Indeed,

$$e^{i\psi} = \frac{(\bar{a}e^{i\theta/2} - be^{-i\theta/2})^2}{|\bar{a}e^{i\theta/2} - be^{-i\theta/2}|^2} = \frac{\bar{a}e^{i\theta/2} - be^{-i\theta/2}}{ae^{-i\theta/2} - \bar{b}e^{i\theta/2}} = \frac{\bar{a}e^{i\theta} - b}{-\bar{b}e^{i\theta} + a}.$$

Since, in addition,

$$|\bar{a}e^{i\theta/2} - be^{-i\theta/2}| = |-\bar{b}e^{i\theta} + a|,$$

it follows that

$$\begin{aligned} (\pi_\chi(g)f)(e^{i\theta}) &= |\bar{a}e^{i\theta/2} - be^{-i\theta/2}|^{2l} \cdot \varphi(e^{i\psi/2}) \\ &= |-\bar{b}e^{i\theta} + a|^{2l} \cdot f(e^{i\psi}) = |-\bar{b}e^{i\theta} + a|^{2l} \cdot f\left(\frac{\bar{a}e^{i\theta} - b}{-\bar{b}e^{i\theta} + a}\right). \end{aligned}$$

Thus we have proved that for $\chi = (l, 0)$ the representation $\pi_\chi(g)$ is realized in the space $\mathcal{C}^\infty(S^1)$ of complex-valued smooth functions on the circle, and is given by the formula

$$(\pi_\chi(g)f)(e^{i\theta}) = |-\bar{b}e^{i\theta} + a|^{2l} \cdot f\left(\frac{\bar{a}e^{i\theta} - b}{-\bar{b}e^{i\theta} + a}\right).$$

One similarly proves that, for $\chi = (l, 1/2)$ the operators $\pi_\chi(g)$ are given by the formula

$$(\pi_\chi(g)f)(e^{i\theta}) = |-\bar{b}e^{i\theta} + a|^{2l-1} \cdot (-\bar{b}e^{i\theta} + a) \cdot f\left(\frac{\bar{a}e^{i\theta} - b}{-\bar{b}e^{i\theta} + a}\right).$$

These two formulas can be replaced by a single expression:

$$(\pi_\chi(g)f)(e^{i\theta}) = (-\bar{b}e^{i\theta} + a)^{l+\varepsilon} \cdot (-be^{-i\theta} + \bar{a})^{l-\varepsilon} \cdot f\left(\frac{\bar{a}e^{i\theta} - b}{-\bar{b}e^{i\theta} + a}\right).$$

8.3 Action of the Lie Algebra

In this subsection we differentiate $(\pi_\chi, \mathcal{C}^\infty(S^1))$ and compute the actions of the Lie algebra $\mathfrak{su}(1, 1)$ and its complexification $\mathfrak{sl}(2, \mathbb{C})$. Note that every element of $\mathcal{C}^\infty(S^1)$ is a smooth vector.

We choose a basis of $\mathfrak{su}(1, 1)$

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} \exp(tX) &= \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, & \exp(tY) &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \\ \exp(tZ) &= \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix}. \end{aligned}$$

We have:

$$(\pi_\chi(\exp(tX))f)(e^{i\theta}) = e^{2\varepsilon it} \cdot f(e^{i(\theta-2t)}),$$

$$\begin{aligned} (\pi_\chi(X)f)(e^{i\theta}) &= \frac{d}{dt}(\pi_\chi(\exp(tX))f)(e^{i\theta}) \Big|_{t=0} \\ &= 2\left(i\varepsilon - \frac{d}{d\theta}\right)f(e^{i\theta}); \end{aligned}$$

$$\begin{aligned} (\pi_\chi(\exp(tY))f)(e^{i\theta}) &= (-\sinh te^{i\theta} + \cosh t)^{l+\varepsilon} \\ &\quad \times (-\sinh te^{-i\theta} + \cosh t)^{l-\varepsilon} \cdot f\left(\frac{\cosh te^{i\theta} - \sinh t}{-\sinh te^{i\theta} + \cosh t}\right), \end{aligned}$$

$$\begin{aligned}
(\pi_\chi(Y)f)(e^{i\theta}) &= \frac{d}{dt}(\pi_\chi(\exp(tY))f)(e^{i\theta}) \Big|_{t=0} \\
&= -\left((l+\varepsilon)e^{i\theta} + (l-\varepsilon)e^{-i\theta} - 2\sin\theta \frac{d}{d\theta}\right)f(e^{i\theta});
\end{aligned}$$

$$\begin{aligned}
(\pi_\chi(\exp(tZ))f)(e^{i\theta}) &= (i\sinh te^{i\theta} + \cosh t)^{l+\varepsilon} \\
&\quad \times (-i\sinh te^{-i\theta} + \cosh t)^{l-\varepsilon} \cdot f\left(\frac{\cosh te^{i\theta} - i\sinh t}{i\sinh te^{i\theta} + \cosh t}\right),
\end{aligned}$$

$$\begin{aligned}
(\pi_\chi(Z)f)(e^{i\theta}) &= \frac{d}{dt}(\pi_\chi(\exp(tZ))f)(e^{i\theta}) \Big|_{t=0} \\
&= \left(i(l+\varepsilon)e^{i\theta} - i(l-\varepsilon)e^{-i\theta} - 2\cos\theta \frac{d}{d\theta}\right)f(e^{i\theta}).
\end{aligned}$$

Finally, we rewrite our results in terms of the more familiar basis of $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathbb{C} \otimes \mathfrak{su}(1, 1)$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since

$$H = -iX, \quad E = \frac{1}{2}(Y - iZ), \quad F = \frac{1}{2}(Y + iZ),$$

we obtain:

$$\begin{aligned}
\pi_\chi(H) &= 2\left(i\frac{d}{d\theta} + \varepsilon\right), \\
\pi_\chi(E) &= e^{-i\theta}\left(i\frac{d}{d\theta} - (l - \varepsilon)\right), \\
\pi_\chi(F) &= -e^{i\theta}\left(i\frac{d}{d\theta} + (l + \varepsilon)\right).
\end{aligned}$$

8.4 The Harish-Chandra Module of $(\pi_\chi, \mathcal{C}^\infty(S^1))$

As usual, we choose the diagonal subgroup

$$K_{\mathbb{R}} = \left\{ k_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}; t \in \mathbb{R} \right\} \simeq SO(2) \simeq U(1)$$

as maximal compact subgroup of $SU(1, 1)$. Then

$$(\pi_\chi(k_t)f)(z) = e^{2\varepsilon it} \cdot f(e^{-2it} \cdot z), \quad z \in S^1.$$

The eigenvectors of this action are functions $z^n = e^{in\theta}$, $n \in \mathbb{Z}$,

$$\pi_\chi(k_t)z^n = e^{2i(\varepsilon-n)t} \cdot z^n.$$

Since every smooth function on S^1 has a Fourier series expansion, we see that $f(e^{i\theta}) \in \mathcal{C}^\infty(S^1)$ is $K_{\mathbb{R}}$ -finite if and only if it is a finite linear combination of $z^n = e^{in\theta}$, $n \in \mathbb{Z}$:

$$\mathcal{C}^\infty(S^1)_{fini} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}z^n = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e^{in\theta}.$$

In particular, each representation (π_χ, V_χ) is admissible.

Pick a basis of $\mathcal{C}^\infty(S^1)_{fini}$

$$v_n = z^{-n} = e^{-in\theta}, \quad n \in \mathbb{Z},$$

and let us find the actions of $\pi_\chi(H)$, $\pi_\chi(E)$, $\pi_\chi(F)$ in that basis.

We have:

$$\pi_\chi(H)v_n = 2\left(i\frac{d}{d\theta} + \varepsilon\right)e^{-in\theta} = 2(n + \varepsilon)e^{-in\theta} = 2(n + \varepsilon)v_n,$$

$$\begin{aligned} \pi_\chi(E)v_n &= e^{-i\theta} \left(i\frac{d}{d\theta} - (l - \varepsilon)\right)e^{-in\theta} \\ &= (n - l + \varepsilon)e^{-i(n+1)\theta} = (n - l + \varepsilon)v_{n+1}, \end{aligned}$$

$$\begin{aligned}
\pi_\chi(F)v_n &= -e^{i\theta} \left(i \frac{d}{d\theta} + (l + \varepsilon) \right) e^{-in\theta} \\
&= -(n + l + \varepsilon) e^{-i(n-1)\theta} = -(n + l + \varepsilon) v_{n-1}.
\end{aligned}$$

Thus we have proved:

Lemma 83. *The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ acts on $\mathcal{C}^\infty(S^1)_{fini}$ as follows:*

$$\begin{aligned}
\pi_\chi(H)v_n &= 2(n + \varepsilon)v_n, \\
\pi_\chi(E)v_n &= (n - l + \varepsilon)v_{n+1}, \\
\pi_\chi(F)v_n &= -(n + l + \varepsilon)v_{n-1}.
\end{aligned}$$

Note that the coefficients $(n \pm l + \varepsilon)$ are never zero unless $l + \varepsilon \in \mathbb{Z}$. This means that the Harish-Chandra module $\mathcal{C}^\infty(S^1)_{fini}$ is irreducible if and only if $l + \varepsilon \notin \mathbb{Z}$. Since a representation is irreducible if and only if its underlying Harish-Chandra module is irreducible, we have proved:

Proposition 84. *The representation $(\pi_\chi, \mathcal{C}^\infty(S^1))$ of $SU(1, 1)$ is irreducible if and only if $l + \varepsilon \notin \mathbb{Z}$.*

In particular, for a generic parameter χ , $(\pi_\chi, \mathcal{C}^\infty(S^1))$ is irreducible.

8.5 Decomposition of $(\pi_\chi, \mathcal{C}^\infty(S^1))$ into Irreducible Components

In this subsection we study how the representation $(\pi_\chi, \mathcal{C}^\infty(S^1))$ of $SU(1, 1)$ decomposes into irreducible components. We have seen that if $l + \varepsilon \notin \mathbb{Z}$, then $(\pi_\chi, \mathcal{C}^\infty(S^1))$ is irreducible. Thus we study the case $l + \varepsilon \in \mathbb{Z}$.

Note that

$$\begin{aligned}\pi_\chi(F)v_{-(l+\varepsilon)} &= \pi_\chi(F)e^{i(l+\varepsilon)\theta} = 0 \quad \text{and} \\ \pi_\chi(E)v_{l-\varepsilon} &= \pi_\chi(E)e^{-i(l-\varepsilon)\theta} = 0.\end{aligned}$$

Define (\mathfrak{g}, K) submodules of $\mathcal{C}^\infty(S^1)_{fini}$

$$\begin{aligned}M^+ &= \mathbb{C}\text{-span of } \{v_n; n \geq -(l+\varepsilon)\} \quad \text{and} \\ M^- &= \mathbb{C}\text{-span of } \{v_n; n \leq l-\varepsilon\}.\end{aligned}$$

Their closures in $\mathcal{C}^\infty(S^1)_{fini}$ are subrepresentations of $(\pi_\chi, \mathcal{C}^\infty(S^1))$.

If $l < 0$, the submodules M^+ and M^- are irreducible, $M^+ \cap M^- = \{0\}$. In the notations of Section 4, as a representation of $\mathfrak{sl}(2, \mathbb{C})$, M^+ is the irreducible lowest weight module V_{-2l} of lowest weight $-2l$. Similarly, as a representation of $\mathfrak{sl}(2, \mathbb{C})$, M^- is the irreducible highest weight module \bar{V}_{2l} of highest weight $2l$. In this case, the complete list of proper (\mathfrak{g}, K) submodules of $\mathcal{C}^\infty(S^1)_{fini}$ is

$$M^+, \quad M^-, \quad M^+ \oplus M^- \text{ (if } (l, \varepsilon) \neq (-1/2, 1/2)).$$

If $(l, \varepsilon) \neq (-1/2, 1/2)$, the quotient module

$$\mathcal{C}^\infty(S^1)_{fini}/(M^+ \oplus M^-)$$

is irreducible, has dimension $-(2l+1)$, and will be denoted by $F_{|2(l+1)|}$, which is consistent with notations of Section 4. We have the following exact sequence of (\mathfrak{g}, K) modules

$$0 \rightarrow M^+ \oplus M^- \rightarrow \mathcal{C}^\infty(S^1)_{fini} \rightarrow F_{|2(l+1)|} \rightarrow 0. \quad (5)$$

Taking closures in $\mathcal{C}^\infty(S^1)$ results in a similar exact sequence of representations of $SU(1, 1)$.

In the exceptional case $l = -1/2$, $\varepsilon = 1/2$ we have only two representations:

$$\mathcal{C}^\infty(S^1)_{fini} = M^+ \oplus M^- \quad (6)$$

and M^+ , M^- are irreducible. The corresponding exact sequence of (\mathfrak{g}, K) modules is

$$0 \rightarrow M^+ \oplus M^- \rightarrow \mathcal{C}^\infty(S^1)_{fini} \rightarrow 0.$$

If $l \geq 0$, then M^+ and M^- have a non-trivial intersection which is irreducible, has dimension $2l + 1$, and will be denoted by F_{2l} , which is consistent with notations of Section 4. The complete list of proper (\mathfrak{g}, K) submodules of $\mathcal{C}^\infty(S^1)_{fini}$ is

$$F_{2l}, \quad M^+, \quad M^-.$$

The quotient module

$$\mathcal{C}^\infty(S^1)_{fini}/F_{2l}$$

is a direct sum of two irreducible modules M^+/F_{2l} and M^-/F_{2l} . As representations of $\mathfrak{sl}(2, \mathbb{C})$, M^+/F_{2l} and M^-/F_{2l} are respectively the irreducible lowest and highest weight modules V_{2l+1} of lowest weight $2l + 1$ and $\bar{V}_{-(2l+1)}$ of highest weight $-(2l + 1)$. We have the following exact sequence of (\mathfrak{g}, K) modules

$$0 \rightarrow F_{2l} \rightarrow \mathcal{C}^\infty(S^1)_{fini} \rightarrow (M^+/F_{2l}) \oplus (M^-/F_{2l}) \rightarrow 0.$$

Taking closures in $\mathcal{C}^\infty(S^1)$ results in a similar exact sequence of representations of $SU(1, 1)$.

As a consequence of this discussion we obtain:

Proposition 85. *Each representation (π_χ, V_χ) is admissible and has finite length.*

We conclude this subsection with the following result:

Theorem 86. *Every irreducible representation of $SU(1, 1)$ is infinitesimally equivalent to an irreducible subrepresentation of (π_χ, V_χ) , for some χ .*

Proof. Since $K \simeq \mathbb{C}^\times$ is connected, the K -action on a (\mathfrak{g}, K) module is completely determined by the \mathfrak{g} -action. We will show later (Corollary 98) that if (π, V) is an admissible irreducible representation of $SU(1, 1)$, then the action of $\mathfrak{sl}(2, \mathbb{C})$ on V_{fini} satisfies condition 2 of Corollary 24. Recall that all irreducible $\mathfrak{sl}(2, \mathbb{C})$ -modules satisfying one of the two equivalent conditions of Corollary 24 were classified in Theorem 33. Note that of these irreducible $\mathfrak{sl}(2, \mathbb{C})$ -modules only those with $\pi(H)$ having integer eigenvalues can possibly arise from (\mathfrak{g}, K) modules (homework). On the other hand, all of these arise as submodules of $\mathcal{C}^\infty(S^1)_{fini}$ for appropriate values of χ . Indeed, finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$ -modules F_d appear when $l = d/2$, $l + \varepsilon \in \mathbb{Z}$. The irreducible lowest weight modules V_λ and highest weight modules \bar{V}_λ , $\lambda \in \mathbb{Z}$, appear as M^+ with $l = -\lambda/2$ and M^- with $l = \lambda/2$, $l + \varepsilon \in \mathbb{Z}$. Finally, all irreducible modules $P(\lambda, \mu)$, $\lambda \in \mathbb{Z}$, appear as $\mathcal{C}^\infty(S^1)_{fini}$ when $l + \varepsilon \notin \mathbb{Z}$ (homework). \square

The following corollary is a precise way of saying when an irreducible representation of $\mathfrak{su}(1, 1)$ “lifts” to a representation of $SU(1, 1)$.

Corollary 87. *An irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module satisfying one of the two equivalent conditions of Corollary 24 appears as $\mathfrak{sl}(2, \mathbb{C})$ acting on the space of $SO(2)$ -finite vectors of some admissible irreducible representation of $SU(1, 1)$ if and only if H has integer eigenvalues.*

8.6 Unitary Representations of $SU(1, 1)$

Let us return to the case $l + \varepsilon \in \mathbb{Z}$, $l < 0$. Let \mathcal{D}_l^- denote the closure of M^- in $\mathcal{C}^\infty(S^1)$; it contains functions

$$e^{-i(l-\varepsilon)\theta} \cdot f(e^{i\theta}), \quad \begin{array}{l} f(z) \text{ is holomorphic on a} \\ \text{neighborhood of } \{z \in \mathbb{C}; |z| \leq 1\}. \end{array}$$

Similarly, let \mathcal{D}_l^+ denote the closure of M^+ in $\mathcal{C}^\infty(S^1)$; it contains functions

$$e^{i(l+\varepsilon)\theta} \cdot f(e^{i\theta}), \quad \begin{array}{l} f(z) \text{ is antiholomorphic on a} \\ \text{neighborhood of } \{z \in \mathbb{C}; |z| \leq 1\}. \end{array}$$

If $l \leq -1$, the representations $(\pi_\chi, \mathcal{D}_l^-)$ and $(\pi_\chi, \mathcal{D}_l^+)$ are called respectively the holomorphic and antiholomorphic discrete series. These names reflect the fact that historically they were first realized in the spaces of holomorphic and antiholomorphic functions. If $l = -1/2$, the representations $(\pi_\chi, \mathcal{D}_l^-)$ and $(\pi_\chi, \mathcal{D}_l^+)$ are called the limits of discrete series.

The following is a complete list of irreducible unitary representations of $SU(1, 1)$ (or, more precisely, representations infinitesimally equivalent to irreducible unitary ones):

- The trivial one-dimensional representation;
- The holomorphic and antiholomorphic discrete series $(\pi_\chi, \mathcal{D}_l^-)$ and $(\pi_\chi, \mathcal{D}_l^+)$, $l = -1, -3/2, -2, \dots$, $l + \varepsilon \in \mathbb{Z}$;
- The limits of discrete series $(\pi_\chi, \mathcal{D}_{-1/2}^-)$ and $(\pi_\chi, \mathcal{D}_{-1/2}^+)$, $l = -1/2$, $\varepsilon = 1/2$;
- The spherical unitary principal series $(\pi_\chi, \mathcal{C}^\infty(S^1))$, $l = -1/2 + i\lambda$, $\lambda \in \mathbb{R}$, $\varepsilon = 0$;

- The nonspherical unitary principal series $(\pi_\chi, \mathcal{C}^\infty(S^1))$, $l = -1/2 + i\lambda$, $\lambda \in \mathbb{R}^\times$, $\varepsilon = 1/2$;
- The complementary series $(\pi_\chi, \mathcal{C}^\infty(S^1))$, $l \in (-1, -1/2) \cup (-1/2, 0)$, $\varepsilon = 0$.

Note that in the case of nonspherical unitary principal series we exclude the value $\chi = (-1/2, 1/2)$ because in this case

$$(\pi_{(-1/2, 1/2)}, \mathcal{C}^\infty(S^1)) = (\pi_{(-1/2, 1/2)}, \mathcal{D}_{-1/2}^-) \oplus (\pi_{(-1/2, 1/2)}, \mathcal{D}_{-1/2}^+).$$

In the case of complementary series we exclude $l = -1/2$ because $(\pi_{(-1/2, 0)}, \mathcal{C}^\infty(S^1))$ has already appeared in the spherical unitary principal series.

Of these unitary representations, the following appear in the decomposition of $L^2(SU(1, 1))$ into irreducible components:

- The holomorphic and antiholomorphic discrete series $(\pi_\chi, \mathcal{D}_l^-)$ and $(\pi_\chi, \mathcal{D}_l^+)$, $l = -1, -3/2, -2, \dots$, $l + \varepsilon \in \mathbb{Z}$;
- The spherical unitary principal series $(\pi_\chi, \mathcal{C}^\infty(S^1))$, $l = -1/2 + i\lambda$, $\lambda \in \mathbb{R}$, $\varepsilon = 0$;
- The nonspherical unitary principal series $(\pi_\chi, \mathcal{C}^\infty(S^1))$, $l = -1/2 + i\lambda$, $\lambda \in \mathbb{R}^\times$, $\varepsilon = 1/2$.

The unitary principal series are sometimes called continuous series, so that $L^2(SU(1, 1))$ has a discrete component and a continuous component.

9 Notes on Schur's Lemma

9.1 Classical Version

Recall the “classical” version of Schur's Lemma:

Lemma 88. *Suppose (π, V) is an irreducible representation of a group $G_{\mathbb{R}}$ (or a Lie algebra $\mathfrak{g}_{\mathbb{R}}$) on a complex finite-dimensional vector space V . If $T : V \rightarrow V$ is a linear map such that T commutes with all $\pi(g)$, $g \in G_{\mathbb{R}}$, (respectively with all $\pi(X)$, $X \in \mathfrak{g}_{\mathbb{R}}$), then T is multiplication by a scalar.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of T . Then $T - \lambda \cdot Id_V$ also commutes with $\pi(g)$, for all $g \in G_{\mathbb{R}}$, (respectively with $\pi(X)$, for all $X \in \mathfrak{g}_{\mathbb{R}}$). Hence the λ -eigenspace of T is a non-zero invariant subspace of V . By irreducibility of V , this λ -eigenspace has to be all of V . Therefore, T is $\lambda \cdot Id_V$. \square

Note that the proof depends on the existence of eigenvalues and eigenvectors of the linear map $T : V \rightarrow V$. But if $\dim V = \infty$, T might not have any eigenvectors (e.g. “shift” operator). So $\dim V < \infty$ is essential for this proof. Here we discuss two infinite-dimensional analogues of Schur’s Lemma.

9.2 Schur’s Lemma for Unitary Representations

Here is a version of Schur’s Lemma for unitary representations.

Proposition 89. *A unitary representation (π, V) of a topological group G on a Hilbert space V is irreducible if and only if the only bounded linear operators on V commuting with all $\pi(g)$, $g \in G$, are the scalar operators.*

Proof. If V is reducible with $U \subset V$ a non-trivial closed invariant subspace, then the orthogonal projection on U is a non-scalar bounded linear operator on V commuting with all $\pi(g)$ ’s.

Conversely, let $T : V \rightarrow V$ be a non-scalar bounded linear operator commuting with all $\pi(g)$ ’s. Since (π, V) is unitary, all

$\pi(g)$, $g \in G$, commute with the adjoint operator T^* as well as self-adjoint operators

$$A = \frac{1}{2}(T + T^*) \quad \text{and} \quad B = \frac{1}{2i}(T - T^*).$$

Since $T = A + iB$, at least one of A , B is not scalar, say A . Using the spectral theorem, we can get a projection P which commutes with all operators commuting with A . Intuitively this involves taking a subset S of $\sigma(A)$ – the spectrum of A – and letting P be the orthogonal projection onto the “direct sum of eigenspaces with eigenvalues in S ”. In particular, P commutes with all $\pi(g)$, $g \in G$. Then $V = \ker P \oplus \text{Im } P$. \square

9.3 Dixmier’s Lemma

Lemma 90. *Let V be a countable dimensional vector space over \mathbb{C} . If $T \in \text{End}(V)$, then there exists a scalar $c \in \mathbb{C}$ such that $T - c \cdot \text{Id}_V$ is not invertible on V .*

Proof. Suppose $T - c \cdot \text{Id}_V$ is invertible for all $c \in \mathbb{C}$. Then $P(T)$ is invertible for all polynomials $P \in \mathbb{C}[x]$, $P \neq 0$. Thus, if $R = P/Q$ is a rational function with $P, Q \in \mathbb{C}[x]$, we can define $R(T)$ by $P(T) \cdot (Q(T))^{-1}$. This rule defines a map $\mathbb{C}(x) \rightarrow \text{End}(V)$. If $v \in V$, $v \neq 0$, and $R \in \mathbb{C}(x)$, $R \neq 0$, then $R(T)v \neq 0$. Hence the map

$$\mathbb{C}(x) \rightarrow V, \quad R \mapsto R(T)v,$$

is injective. Since $\mathbb{C}(x)$ has uncountable dimension over \mathbb{C} (for example, the rational functions $(x - a)^{-1}$, $a \in \mathbb{C}$, are linearly independent), we get a contradiction. \square

Example 91. Let V be a countable dimensional vector space with basis $\{e_n\}$, $n \in \mathbb{Z}$. (Thus $V = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}e_n$.) Consider the shift operator $T : V \rightarrow V$ defined by $T(e_n) = e_{n+1}$. Then $T - c \cdot \text{Id}_V$ is not invertible on V whenever $c \in \mathbb{C}^\times$. We have: $\ker(T - c \cdot \text{Id}_V) = 0$ (i.e. T has no eigenvectors) and $\text{Im}(T - c \cdot \text{Id}_V) \subsetneq V$ for all $c \in \mathbb{C}^\times$. For example, if $c = 1$,

$$\text{Im}(T - \text{Id}_V) = \left\{ v = \sum_n a_n e_n \in V; \sum_n a_n = 0 \right\}.$$

Definition 92. Let V be a vector space over \mathbb{C} and $S \subset \text{End}(V)$ a subset. Then S acts irreducibly if whenever $W \subset V$ is a subspace such that $SW \subset W$, then $W = V$ or $W = \{0\}$.

Lemma 93 (Dixmier). Suppose V is countable dimensional and that $S \subset \text{End}(V)$ acts irreducibly. If $T \in \text{End}(V)$ commutes with every element of S , then T is a scalar multiple of the identity operator.

Proof. By the above lemma, there exists a $c \in \mathbb{C}$ such that $T - c \cdot \text{Id}_V$ is not invertible on V . Then $\ker(T - c \cdot \text{Id}_V)$ and $\text{Im}(T - c \cdot \text{Id}_V)$ are preserved by S , hence these spaces are either $\{0\}$ or V . If $\ker(T - c \cdot \text{Id}_V) = \{0\}$ and $\text{Im}(T - c \cdot \text{Id}_V) = V$, then T is invertible (there is no topology or continuity involved here). Hence $\ker(T - c \cdot \text{Id}_V) = V$ or $\text{Im}(T - c \cdot \text{Id}_V) = \{0\}$ and in both cases $T = c \cdot \text{Id}_V$. \square

Note that if V is an admissible (\mathfrak{g}, K) -module, then V has countable dimension, since the set $\hat{K}_{\mathbb{R}}$ is countable. The following lemma essentially tells us that we can drop the admissibility assumption if V is irreducible.

Lemma 94. *Let V be an irreducible (\mathfrak{g}, K) -module, then V has countable dimension.*

Proof. Pick $v \in V$, $v \neq 0$, and define

$$W_v = \mathbb{C}\text{-span of } \{kv; k \in K\}, \quad \dim W_v < \infty.$$

Then $\mathcal{U}(\mathfrak{g})W_v$ is a (\mathfrak{g}, K) -invariant subspace of V of countable dimension (the last statement follows from the Poincaré-Birkhoff-Witt Theorem). Since V is irreducible, $V = \mathcal{U}(\mathfrak{g})W_v$ and V has countable dimension. \square

Corollary 95. *Let V be an irreducible (\mathfrak{g}, K) -module, then*

$$\mathrm{Hom}_{(\mathfrak{g}, K)}(V, V) = \mathbb{C} \cdot \mathrm{Id}_V.$$

Proof. Dixmier's Lemma applies, since V has countable dimension. \square

9.4 Infinitesimal Characters

Let $\mathcal{Z}\mathcal{U}(\mathfrak{g})$ denote the center of $\mathcal{U}(\mathfrak{g})$ and introduce

$$\mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}}) =_{\mathrm{def}} \{a \in \mathcal{U}(\mathfrak{g}); \mathrm{Ad}(g)a = a, \forall g \in G_{\mathbb{R}}\}.$$

Clearly, $\mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}}) \subset \mathcal{Z}\mathcal{U}(\mathfrak{g})$ and

$$\mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}}) = \mathcal{Z}\mathcal{U}(\mathfrak{g}) \quad \text{if } G_{\mathbb{R}} \text{ is connected.}$$

Applying Dixmier's Lemma we obtain:

Lemma 96. *Let V be an irreducible (\mathfrak{g}, K) -module. Then each $z \in \mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}})$ acts on V by a scalar multiple of identity.*

Corollary 97. *Let (π, V) be an irreducible admissible representation of $G_{\mathbb{R}}$, then each $z \in \mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}})$ acts on V^{∞} by a scalar multiple of identity.*

Proof. Apply the above lemma to V_{fini} , which is dense in V^∞ . \square

Corollary 98. *Let (π, V) be an irreducible admissible representation of $SU(1, 1)$, then the Casimir element acts on V_{fini} by multiplication by a scalar. In particular, the action of $\mathfrak{sl}(2, \mathbb{C})$ on V_{fini} satisfies condition 2 of Corollary 24.*

Definition 99. *If (π, V) is a representation of $G_{\mathbb{R}}$ and χ is a homomorphism of $\mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}})$ into \mathbb{C} such that*

$$\pi(z)v = \chi(z)v, \quad \forall z \in \mathcal{Z}_{\mathcal{U}(\mathfrak{g})}(G_{\mathbb{R}}), v \in V^\infty,$$

then χ is called the infinitesimal character of (π, V) .

By Corollary 97, every admissible irreducible representation of $G_{\mathbb{R}}$ has an infinitesimal character.

Caution: Even if $G_{\mathbb{R}}$ is connected, some non-equivalent irreducible representations may have the same infinitesimal character. For example, representations (π_χ, V_χ) of $SU(1, 1)$ always have infinitesimal character. Some of these are reducible, and their irreducible components have the same infinitesimal character.

10 Iwasawa and KAK Decompositions

In this section we describe Iwasawa and KAK decompositions. For proofs and details see the book [13].

10.1 Iwasawa Decomposition

An important class of representations are the so-called principal series representations. The Subrepresentation Theorem

says that every irreducible admissible representation of $G_{\mathbb{R}}$ is infinitesimally equivalent to a subrepresentation of some principal series representation. To construct these we will need Iwasawa decomposition, which is sometimes called the KAN decomposition.

Recall the Cartan involution and the Cartan decomposition:

$$\begin{aligned}\theta : \mathfrak{g}_{\mathbb{R}} &\rightarrow \mathfrak{g}_{\mathbb{R}}, & \theta^2 &= Id, \\ \mathfrak{g}_{\mathbb{R}} &= \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}, & \theta|_{\mathfrak{k}_{\mathbb{R}}} &= Id, & \theta|_{\mathfrak{p}_{\mathbb{R}}} &= -Id, \\ [\mathfrak{k}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}] &\subset \mathfrak{k}_{\mathbb{R}}, & [\mathfrak{p}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] &\subset \mathfrak{k}_{\mathbb{R}}, & [\mathfrak{k}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] &\subset \mathfrak{p}_{\mathbb{R}}.\end{aligned}$$

Let $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ be the connected Lie subgroup with Lie algebra $\mathfrak{k}_{\mathbb{R}}$, then $K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$, and it acts on $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$ by Ad . Every $X \in \mathfrak{p}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is diagonalizable with real eigenvalues, hence the action of $\text{ad } X$ on $\mathfrak{g}_{\mathbb{R}}$ is diagonalizable and has real eigenvalues as well.

Choose a maximal abelian subspace $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{p}_{\mathbb{R}}$. Clearly, $\dim \mathfrak{a}_{\mathbb{R}} \geq 1$, $\mathfrak{a}_{\mathbb{R}}$ is θ -invariant and, for all $X \in \mathfrak{a}_{\mathbb{R}}$, $\text{ad } X$ acts on $\mathfrak{g}_{\mathbb{R}}$ semisimply with real eigenvalues. Hence

$$Z_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}}) = \{Y \in \mathfrak{g}_{\mathbb{R}}; [Y, X] = 0, \forall X \in \mathfrak{a}_{\mathbb{R}}\} = \mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}},$$

where

$$\mathfrak{m}_{\mathbb{R}} = Z_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}}) = \{Y \in \mathfrak{k}_{\mathbb{R}}; [Y, X] = 0, \forall X \in \mathfrak{a}_{\mathbb{R}}\}.$$

Let

$$\begin{aligned}A_{\mathbb{R}} &=_{\text{def}} \text{connected subgroup of } G_{\mathbb{R}} \text{ with Lie algebra } \mathfrak{a}_{\mathbb{R}}, \\ M_{\mathbb{R}} &=_{\text{def}} Z_{K_{\mathbb{R}}}(A_{\mathbb{R}}) = Z_{K_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}}).\end{aligned}$$

The group $M_{\mathbb{R}}$ need not be connected, and so $M_{\mathbb{R}} \neq \exp \mathfrak{m}_{\mathbb{R}}$ in general, but its Lie algebra is $\mathfrak{m}_{\mathbb{R}}$. Complexify:

$$\mathfrak{a} =_{\text{def}} \mathbb{C} \otimes \mathfrak{a}_{\mathbb{R}}, \quad \mathfrak{m} =_{\text{def}} \mathbb{C} \otimes \mathfrak{m}_{\mathbb{R}}, \quad \mathfrak{a}, \mathfrak{m} \subset \mathfrak{g} = \mathbb{C} \otimes \mathfrak{g}_{\mathbb{R}}.$$

Consider the adjoint action of \mathfrak{a} on \mathfrak{g} (we already know that it is semisimple). We have:

$$Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a} \quad \text{and} \quad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\alpha} \right),$$

where $\Phi = \Phi(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^* \setminus \{0\}$ is the “reduction of relative roots of the pair $(\mathfrak{g}, \mathfrak{a})$ ” and

$$\mathfrak{g}^{\alpha} = \{Y \in \mathfrak{g}; (\text{ad } X)Y = \alpha(X) \cdot Y, \forall X \in \mathfrak{a}\}.$$

(Caution: \mathfrak{a} need not be a Cartan subalgebra of \mathfrak{g} , but it is a somewhat similar object.) Both θ and complex conjugation (with respect to $\mathfrak{g}_{\mathbb{R}}$) act on \mathfrak{g} . We know that $\theta = -Id$ on \mathfrak{a} and the roots are real-valued on $\mathfrak{a}_{\mathbb{R}}$. Therefore, for all $\alpha \in \Phi$,

$$\theta\alpha = -\alpha, \quad \theta(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{-\alpha}, \quad \overline{\mathfrak{g}^{\alpha}} = \mathfrak{g}^{\alpha}.$$

All properties of root space decomposition of a complex semisimple Lie algebra carry over to this setting, except

1. $\dim \mathfrak{g}^{\alpha}$ can be strictly bigger than 1;
2. It is possible to have non-trivially proportional roots $\pm\alpha$ and $\pm 2\alpha$ only.

(This is why non-reduced root systems are important.)

A *Weyl chamber* is a connected component of

$$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Phi(\mathfrak{g}, \mathfrak{a})} \{X \in \mathfrak{a}_{\mathbb{R}}; \alpha(X) = 0\}$$

($\mathfrak{a}_{\mathbb{R}}$ without a finite number of hyperplanes). Each Weyl chamber is a convex cone. Choose one and call it the “highest Weyl chamber”. Then there exists a uniquely determined subset $\Phi^+ \subset \Phi(\mathfrak{g}, \mathfrak{a})$ such that the highest Weyl chamber is precisely

$$\{X \in \mathfrak{a}_{\mathbb{R}}; \alpha(X) > 0 \forall \alpha \in \Phi^+\}$$

Consider an analogue of the Weyl group

$$W = W(G_{\mathbb{R}}, A_{\mathbb{R}}) = N_{K_{\mathbb{R}}}(A_{\mathbb{R}})/Z_{K_{\mathbb{R}}}(A_{\mathbb{R}}) = N_{K_{\mathbb{R}}}(A_{\mathbb{R}})/M_{\mathbb{R}}.$$

This is a finite group generated by reflections about root hyperplanes, it acts simply transitively on the set of Weyl chambers.

Define

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}, \quad \mathfrak{n} \subset \mathfrak{g},$$

this is a nilpotent subalgebra defined over \mathbb{R} , i.e. if we let $\mathfrak{n}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{n}$, then $\mathfrak{n} = \mathbb{C} \otimes \mathfrak{n}_{\mathbb{R}}$. Let

$$\begin{aligned} N_{\mathbb{R}} &= \text{connected subgroup of } G_{\mathbb{R}} \text{ with Lie algebra } \mathfrak{n}_{\mathbb{R}} \\ &= \exp(\mathfrak{n}_{\mathbb{R}}). \end{aligned}$$

Note that $\exp : \mathfrak{n}_{\mathbb{R}} \rightarrow N_{\mathbb{R}}$ is a polynomial map.

Theorem 100 (Iwasawa Decomposition). a) *Any two maximal abelian subspaces of $\mathfrak{p}_{\mathbb{R}}$ are conjugate under $K_{\mathbb{R}}$;*
b) *$M_{\mathbb{R}}$, $A_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are closed subgroups of $G_{\mathbb{R}}$, the groups $A_{\mathbb{R}}$ and $M_{\mathbb{R}}$ commute, both normalize $N_{\mathbb{R}}$;*
c) *The map $K_{\mathbb{R}} \times A_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow G_{\mathbb{R}}$, $(k, a, n) \mapsto kan$, is a diffeomorphism.*

(Sometimes this decomposition is called the *KAN* decomposition.)

Example 101. Let $G_{\mathbb{R}} = SL(n, \mathbb{R})$, $K_{\mathbb{R}} = SO(n)$,

$$\mathfrak{k}_{\mathbb{R}} = \{X \in \text{End}(\mathbb{R}^n); X^T = -X, \text{Tr } X = 0\},$$

$$\mathfrak{p}_{\mathbb{R}} = \{X \in \text{End}(\mathbb{R}^n); X^T = X, \text{Tr } X = 0\},$$

$$\mathfrak{a}_{\mathbb{R}} = \left\{ X \in \text{End}(\mathbb{R}^n); \begin{array}{l} X \text{ is diagonal with respect to} \\ \text{the standard basis and } \text{Tr } X = 0 \end{array} \right\},$$

$$A_{\mathbb{R}} = \left\{ a \in SL(n, \mathbb{R}); \begin{array}{l} a \text{ is diagonal with} \\ \text{positive diagonal entries} \end{array} \right\},$$

$$M_{\mathbb{R}} = \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & \varepsilon_n \end{pmatrix} ; \varepsilon_j \in \{\pm 1\}, \prod_j \varepsilon_j = 1 \right\}.$$

In this case, $\dim M_{\mathbb{R}} = 0$, $\mathfrak{m}_{\mathbb{R}} = 0$ and $\mathfrak{a}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$ (meaning that \mathfrak{a} is a Cartan subalgebra of \mathfrak{g}). This implies that $\Phi(\mathfrak{g}_{\mathbb{R}}, \mathfrak{a}_{\mathbb{R}}) = \Phi(\mathfrak{g}, \mathfrak{a})$ is the full root system of $\mathfrak{sl}(n, \mathbb{C})$. In particular, there are no non-trivially proportional roots, $\dim \mathfrak{g}^{\alpha} = 1$ for all $\alpha \in \Phi$ and W is the full Weyl group. This situation is referred to by saying that $G_{\mathbb{R}}$ is split. Each complex semisimple Lie group has a split real form, which is unique up to conjugation. One can think of it as the opposite extreme to the compact real form. For example, $SL(n, \mathbb{C})$ has real forms: compact $SU(n)$, split $SL(n, \mathbb{R})$ as well as others (such as $SU(p, q)$, $p + q = n$) if $n > 2$. Write

$$\mathfrak{a}_{\mathbb{R}} = \left\{ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & x_n \end{pmatrix} ; x_j \in \mathbb{R}, \sum_j x_j = 0 \right\}.$$

The group W acting on $\mathfrak{a}_{\mathbb{R}}$ is isomorphic to the symmetric group S_n acting on $\mathfrak{a}_{\mathbb{R}}$ by permutation of the x_j 's. The root

system is

$$\Phi = \{\pm(x_j^* - x_k^*); 1 \leq j < k \leq n\},$$

where $x_1^*, \dots, x_n^* \in (\mathbb{R}^n)^*$ denote the dual basis of \mathbb{R}^n so that $\langle x_j^*, x_k \rangle = \delta_{jk}$. We can choose the highest Weyl chamber to be

$$\left\{ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & x_n \end{pmatrix} ; x_j \in \mathbb{R}, x_1 > x_2 > \cdots > x_n, \sum_j x_j = 0 \right\},$$

then

$$\Phi^+ = \{x_j^* - x_k^*; 1 \leq j < k \leq n\},$$

$\mathfrak{n}_{\mathbb{R}} =$ Lie algebra of strictly lower-triangular matrices,

$N_{\mathbb{R}} =$ Lie group of lower-triangular unipotent matrices.

The Iwasawa decomposition for $GL(n, \mathbb{R})$ (requires a slight modification for $SL(n, \mathbb{R})$) asserts that any invertible $n \times n$ real matrix can be expressed uniquely as

$$\begin{pmatrix} \text{orthogonal} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \text{diagonal matrix with} \\ \text{positive entries} \end{pmatrix} \begin{pmatrix} \text{lower unipotent} \\ \text{matrix} \end{pmatrix}$$

and these three factors are real-analytic functions. This can be proved using the Gramm-Schmidt orthogonalization process, and the general case can be reduced to $GL(n, \mathbb{R})$.

Recall that by Theorem 100 the groups $M_{\mathbb{R}}, A_{\mathbb{R}}$ normalize $N_{\mathbb{R}}$, $A_{\mathbb{R}} \cdot N_{\mathbb{R}}$ is closed in $G_{\mathbb{R}}$, $M_{\mathbb{R}}$ is compact. Hence

$$P_{\mathbb{R}} =_{\text{def}} M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$$

is a closed subgroup of $G_{\mathbb{R}}$ called the *minimal parabolic subgroup*. Then $N_{\mathbb{R}} \subset P_{\mathbb{R}}$ is a closed normal subgroup.

Caution: Beware of the following clash of notations. The Lie algebra of $P_{\mathbb{R}}$ is $\mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} + \mathfrak{n}_{\mathbb{R}}$ and not the subspace $\mathfrak{p}_{\mathbb{R}}$ from the Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$. Unless, $[\mathfrak{p}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}] = 0$, $\mathfrak{p}_{\mathbb{R}}$ is not even a Lie algebra.

Lemma 102. *We have: $P_{\mathbb{R}} \cap K_{\mathbb{R}} = M_{\mathbb{R}}$.*

Proof. Since $M_{\mathbb{R}} \subset P_{\mathbb{R}} \cap K_{\mathbb{R}}$, it is enough to show that $K_{\mathbb{R}} \cap (A_{\mathbb{R}}N_{\mathbb{R}}) = \{e\}$, but this follows from Iwasawa decomposition. \square

From this lemma and Iwasawa decomposition we immediately obtain:

Corollary 103. *The homogeneous space $G_{\mathbb{R}}/P_{\mathbb{R}} \simeq K_{\mathbb{R}}/M_{\mathbb{R}}$ is compact.*

10.2 KAK Decomposition

By part (a) of Theorem 100,

$$\mathfrak{p}_{\mathbb{R}} = \bigcup_{k \in K_{\mathbb{R}}} \text{Ad}(k)\mathfrak{a}_{\mathbb{R}}.$$

Then it is immediate from the global Cartan decomposition $G_{\mathbb{R}} = K_{\mathbb{R}} \cdot \exp \mathfrak{p}_{\mathbb{R}}$ that $G_{\mathbb{R}} = K_{\mathbb{R}}A_{\mathbb{R}}K_{\mathbb{R}}$ in the sense that every element $g \in G_{\mathbb{R}}$ can be expressed as $g = k_1ak_2$ with $k_1, k_2 \in K_{\mathbb{R}}$ and $a \in A_{\mathbb{R}}$. The following theorem describes the degree of non-uniqueness of this decomposition.

Theorem 104 (*KAK Decomposition*). *Every element $g \in G_{\mathbb{R}}$ can be expressed as $g = k_1ak_2$ with $k_1, k_2 \in K_{\mathbb{R}}$ and $a \in A_{\mathbb{R}}$. In this decomposition, a is uniquely determined up to conjugation by an element of $W(G_{\mathbb{R}}, A_{\mathbb{R}})$. If a is fixed as*

$\exp X$ with $X \in \mathfrak{a}_{\mathbb{R}}$ and if $\alpha(X) \neq 0$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{a})$, then k_1 is unique up to multiplication on the right by an element of $M_{\mathbb{R}}$.

Note that the roles of k_1 and k_2 in this decomposition are symmetric, since $g^{-1} = (k_2)^{-1}a^{-1}(k_1)^{-1}$. Hence, if a is fixed as $\exp X$ with $X \in \mathfrak{a}_{\mathbb{R}}$ and if $\alpha(X) \neq 0$ for all $\alpha \in \Phi(\mathfrak{g}, \mathfrak{a})$, then k_2 is unique up to multiplication on the left by an element of $M_{\mathbb{R}}$.

11 Principal Series Representations

11.1 $G_{\mathbb{R}}$ -equivariant Vector Bundles

The principal series representations of $G_{\mathbb{R}}$ are realized in the space of sections of certain $G_{\mathbb{R}}$ -equivariant vector bundles over a homogeneous space $G_{\mathbb{R}}/P_{\mathbb{R}}$. For this reason we review equivariant vector bundles.

Let G be any Lie group and suppose that it acts on a manifold M . For concreteness, let us suppose that the manifold, the action map and the vector bundles are all smooth. But keep in mind that everything we do in this subsection applies to the complex analytic setting as well.

A G -equivariant vector bundle on M is a vector bundle $\mathcal{E} \rightarrow M$ with G -action on \mathcal{E} by bundle maps lying over the G -action on M . Write \mathcal{E}_m for the fiber of \mathcal{E} at $m \in M$, then, for each $g \in G$,

$g : \mathcal{E}_m \rightarrow \mathcal{E}_{g \cdot m}$ is a linear isomorphism of vector spaces.

Now suppose that G acts on M transitively. This means that M is a homogeneous space G/H for some closed subgroup

$H \subset G$. We have a bijection:

$$\left\{ \begin{array}{l} G\text{-equivariant complex vector} \\ \text{bundles over } G/H \text{ of rank } n \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{complex representations} \\ \text{of } H \text{ of dimension } n \end{array} \right\}$$

In one direction, if $\mathcal{E} \rightarrow G/H$ is a G -equivariant vector complex bundle of rank n , then H acts on the n -dimensional vector space $E = \mathcal{E}_{eH}$ linearly, hence we get a representation (τ, E) of H . Moreover, we have an isomorphism

$$\begin{array}{ccc} \mathcal{E} & \simeq & G \times_H E =_{\text{def}} G \times E / \sim \\ \downarrow & & \downarrow \\ G/H & = & G/H \end{array} \quad (gh, v) \sim (g, \tau(h)v). \quad (7)$$

To see this, note that the map

$$G \times \mathcal{E}_{eH} \rightarrow \mathcal{E}, \quad (g, v) \mapsto gv \in \mathcal{E}_{gH} \subset \mathcal{E},$$

is onto and then determine which points get identified with which. Conversely, any representation (τ, E) of H determines a G -equivariant vector bundle by (7), $\text{rank } \mathcal{E} = \dim E$. And, of course, we have a similar bijection between real vector bundles over G/H and real representations of H .

Now, let $\mathcal{E} \rightarrow G/H$ be the vector bundle corresponding to a representation (τ, E) of H . Then the global sections of \mathcal{E} can be described as follows:

$$\mathcal{C}^\infty(G/H, \mathcal{E}) \simeq$$

$$\{f : G \xrightarrow{\text{smooth}} E; f(gh) = \tau(h^{-1})f(g), \forall g \in G, h \in H\}$$

(because $(g, v) = (gh, \tau(h^{-1})v)$ in $G \times_H E$).

11.2 Construction of the Principal Series Representations

In this subsection we construct the principal series representations. These are obtained by *inducing* representations from

$P_{\mathbb{R}}$ to $G_{\mathbb{R}}$. In other words, one starts with a finite-dimensional representation of $P_{\mathbb{R}}$, forms the corresponding $G_{\mathbb{R}}$ -equivariant vector bundle $\mathcal{E} \rightarrow G_{\mathbb{R}}/P_{\mathbb{R}}$ and lets $G_{\mathbb{R}}$ act on $\mathcal{C}^{\infty}(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E})$ – the space of global sections of that bundle. When the group $G_{\mathbb{R}}$ is $SU(1, 1)$, the principal series representations are precisely the representations (π_{χ}, V_{χ}) constructed in Section 8. We use the same notations as in Subsection 10.1 where we discussed the Iwasawa decomposition.

Note that any $\nu \in \mathfrak{a}^*$ lifts to a multiplicative character (i.e. one-dimensional representation)

$$e^{\nu} : A_{\mathbb{R}} \rightarrow \mathbb{C}^{\times} \quad \text{by the rule} \quad e^{\nu}(\exp X) =_{\text{def}} e^{\nu(X)}, \quad X \in \mathfrak{a}_{\mathbb{R}},$$

then

$$e^{\nu}(a_1) \cdot e^{\nu}(a_2) = e^{\nu}(a_1 a_2), \quad \forall a_1, a_2 \in A_{\mathbb{R}}.$$

Using the decomposition $P_{\mathbb{R}} = M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$, we can extend e^{ν} to a one-dimensional representation $e^{\nu} : P_{\mathbb{R}} \rightarrow \mathbb{C}^{\times}$ with e^{ν} being trivial on $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$:

$$e^{\nu}(man) = e^{\nu}(a), \quad m \in M_{\mathbb{R}}, a \in A_{\mathbb{R}}, n \in N_{\mathbb{R}}.$$

Note that, since $M_{\mathbb{R}}$ and $A_{\mathbb{R}}$ normalize $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ commutes with $A_{\mathbb{R}}$, it is true that

$$e^{\nu}(p_1) \cdot e^{\nu}(p_2) = e^{\nu}(p_1 p_2), \quad \forall p_1, p_2 \in P_{\mathbb{R}}.$$

This one-dimensional representation determines a $G_{\mathbb{R}}$ -equivariant line bundle $\mathcal{L}_{\nu} \rightarrow G_{\mathbb{R}}/P_{\mathbb{R}}$.

In particular, we can apply above construction to $\nu = -2\rho$, where

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} (\dim \mathfrak{g}^{\alpha}) \alpha \in \mathfrak{a}_{\mathbb{R}}^* \subset \mathfrak{a}^*,$$

and obtain a line bundle $\mathcal{L}_{-2\rho} \rightarrow G_{\mathbb{R}}/P_{\mathbb{R}}$. Since $\rho \in \mathfrak{a}_{\mathbb{R}}^*$, this line bundle has a real structure, i.e. can be obtained by complexifying a certain $G_{\mathbb{R}}$ -equivariant *real* line bundle.

Lemma 105. *As a $G_{\mathbb{R}}$ -equivariant line bundle with real structure, $\mathcal{L}_{-2\rho} \rightarrow G_{\mathbb{R}}/P_{\mathbb{R}}$ is isomorphic to the top exterior power of the complexified cotangent bundle of $G_{\mathbb{R}}/P_{\mathbb{R}}$, $\mathbb{C} \otimes \Lambda^{\text{top}} T^* G_{\mathbb{R}}/P_{\mathbb{R}}$.*

Proof. Since the tangent space $G_{\mathbb{R}}/P_{\mathbb{R}}$ at $eP_{\mathbb{R}}$ is $\mathfrak{g}_{\mathbb{R}}/\text{Lie}(P_{\mathbb{R}})$, the cotangent space at $eP_{\mathbb{R}}$ is the annihilator of $\text{Lie}(P_{\mathbb{R}})$ in $\mathfrak{g}_{\mathbb{R}}^*$:

$$T_{eP_{\mathbb{R}}}^* G_{\mathbb{R}}/P_{\mathbb{R}} = (\mathfrak{g}_{\mathbb{R}}/\text{Lie}(P_{\mathbb{R}}))^* = \{l \in \mathfrak{g}_{\mathbb{R}}^*; l|_{\text{Lie}(P_{\mathbb{R}})} = 0\}.$$

As an $M_{\mathbb{R}}A_{\mathbb{R}}$ -module, $T_{eP_{\mathbb{R}}} G_{\mathbb{R}}/P_{\mathbb{R}}$ is

$$\mathfrak{g}_{\mathbb{R}}/\text{Lie}(P_{\mathbb{R}}) = (\mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}} \oplus \theta\mathfrak{n}_{\mathbb{R}})/(\mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}}) = \theta\mathfrak{n}_{\mathbb{R}}.$$

Identifying $\mathfrak{g}_{\mathbb{R}}$ with its dual $\mathfrak{g}_{\mathbb{R}}^*$ via the Killing form and noting that $\mathfrak{n}_{\mathbb{R}}$ and $\theta\mathfrak{n}_{\mathbb{R}}$ are dual to each other, we conclude that

$$T_{eP_{\mathbb{R}}}^* G_{\mathbb{R}}/P_{\mathbb{R}} \simeq \mathfrak{n}_{\mathbb{R}}$$

as $M_{\mathbb{R}}A_{\mathbb{R}}$ -modules. Then we have an isomorphism of $P_{\mathbb{R}}$ -modules

$$\Lambda^{\text{top}} T_{eP_{\mathbb{R}}}^* G_{\mathbb{R}}/P_{\mathbb{R}} \simeq \Lambda^{\text{top}} \mathfrak{n}_{\mathbb{R}},$$

since $N_{\mathbb{R}}$ acts trivially on both sides.

Since we have a bijection between $G_{\mathbb{R}}$ -equivariant complex line bundles over $G_{\mathbb{R}}/P_{\mathbb{R}}$ and one-dimensional complex representations of $P_{\mathbb{R}}$, it is sufficient to show isomorphism of $P_{\mathbb{R}}$ -modules

$$\Lambda^{\text{top}} T_{eP_{\mathbb{R}}}^* G_{\mathbb{R}}/P_{\mathbb{R}} \simeq \Lambda^{\text{top}} \mathfrak{n}_{\mathbb{R}} \quad \text{and} \quad (e^{-2\rho}, \mathbb{R}).$$

It is certainly clear that the actions of $A_{\mathbb{R}}$ and $N_{\mathbb{R}}$ on both sides are the same. Since the trivial representation is the only one-dimensional representation of a compact connected Lie group,

$M_{\mathbb{R}}^0$ – the connected component of the identity element in $M_{\mathbb{R}}$ – acts trivially on both sides. It remains to show that every connected component of $M_{\mathbb{R}}$ contains a representative m which acts trivially on $\Lambda^{\text{top}}\mathfrak{n}_{\mathbb{R}}$. This part requires a bit more work and we omit it. \square

Corollary 106. *The homogeneous space $G_{\mathbb{R}}/P_{\mathbb{R}}$ is orientable.*

Proof. Recall that a manifold M is orientable if and only if the top exterior power of the cotangent bundle $\Lambda^{\text{top}}T^*M$ has a non-vanishing global section. In our case $M = G_{\mathbb{R}}/P_{\mathbb{R}}$ and $\Lambda^{\text{top}}T^*M$ is isomorphic to the real subbundle of $\mathcal{L}_{-2\rho}$. The group $K_{\mathbb{R}}$ acts on $G_{\mathbb{R}}/P_{\mathbb{R}} \simeq K_{\mathbb{R}}/M_{\mathbb{R}}$ transitively, and it is easy to see that there is a real $K_{\mathbb{R}}$ -invariant non-vanishing section of $\mathcal{L}_{-2\rho}$ – the one corresponding to a nonzero constant map $K_{\mathbb{R}} \rightarrow \mathbb{R}$. \square

Now we have everything we need to construct the principal series representations of $G_{\mathbb{R}}$. Let (τ, E) be an irreducible representation of $M_{\mathbb{R}}$ (over \mathbb{C}), and pick an element $\nu \in \mathfrak{a}^*$. Define a representation (τ_{ν}, E) of $P_{\mathbb{R}}$ as follows. Informally, it can be described by saying that $A_{\mathbb{R}}$ acts on E by $e^{\nu-\rho} \cdot \text{Id}_E$, $M_{\mathbb{R}}$ acts by τ and $N_{\mathbb{R}}$ acts trivially. More precisely, we first consider a representation of $M_{\mathbb{R}} \times A_{\mathbb{R}}$ which is the exterior tensor product representation $\tau \boxtimes e^{\nu-\rho}$ on $E \otimes \mathbb{C} \simeq E$, and then define the representation (τ_{ν}, E) of $P_{\mathbb{R}}$ by extending $\tau \boxtimes e^{\nu-\rho}$ via the isomorphism $P_{\mathbb{R}}/N_{\mathbb{R}} \simeq M_{\mathbb{R}}A_{\mathbb{R}} \simeq M_{\mathbb{R}} \times A_{\mathbb{R}}$.

By above construction, the representation (τ_{ν}, E) leads to a $G_{\mathbb{R}}$ -equivariant vector bundle $\mathcal{E}_{\nu} \rightarrow G_{\mathbb{R}}/P_{\mathbb{R}}$. Its global sections

are

$$\mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu) \simeq$$

$$\{f : G_{\mathbb{R}} \xrightarrow{\mathcal{C}^\infty} E; f(gp) = \tau_\nu(p^{-1})f(g), \forall g \in G_{\mathbb{R}}, p \in P_{\mathbb{R}}\}.$$

The group $G_{\mathbb{R}}$ acts on $\mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu)$ by left translations; we denote this action by $\pi_{E,\nu}$ and call it a *principal series representation*:

$$(\pi_{E,\nu}(\tilde{g})f)(g) = f(\tilde{g}^{-1}g), \quad \tilde{g}, g \in G_{\mathbb{R}}, f \in \mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu).$$

11.3 Admissibility of the Principal Series Representations

Recall that we have a diffeomorphism of $K_{\mathbb{R}}$ -spaces $G_{\mathbb{R}}/P_{\mathbb{R}} \simeq K_{\mathbb{R}}/M_{\mathbb{R}}$. This induces isomorphisms of vector spaces:

$$\begin{aligned} & \mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu) \\ & \simeq \{f : G_{\mathbb{R}} \xrightarrow{\mathcal{C}^\infty} E; f(gp) = \tau_\nu(p^{-1})f(g), \forall g \in G_{\mathbb{R}}, p \in P_{\mathbb{R}}\} \\ & \simeq \{f : K_{\mathbb{R}} \xrightarrow{\mathcal{C}^\infty} E; f(km) = \tau(m^{-1})f(k), \forall k \in K_{\mathbb{R}}, m \in M_{\mathbb{R}}\} \\ & \simeq \mathcal{C}^\infty(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_\nu). \quad (8) \end{aligned}$$

The last two spaces are (in obvious way) only $K_{\mathbb{R}}$ -modules.

We would like to know how $\pi_{E,\nu}$ acts on $\mathcal{C}^\infty(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_\nu)$. For an element $g \in G_{\mathbb{R}}$, let $k(g) \in K_{\mathbb{R}}$, $a(g) \in A_{\mathbb{R}}$ and $n(g) \in N_{\mathbb{R}}$ be the Iwasawa components of g . Suppose that $f : K_{\mathbb{R}} \rightarrow E$ is a smooth function such that

$$f(um) = \tau(m^{-1})f(u), \quad \forall u \in K_{\mathbb{R}}, m \in M_{\mathbb{R}}.$$

We can think of f as a function on $G_{\mathbb{R}}$ via the isomorphism (8), then, for all $g \in G_{\mathbb{R}}$,

$$\begin{aligned} (\pi_{E,\nu}(g)f)(u) &= f(g^{-1}u) = f(k(g^{-1}u) \cdot a(g^{-1}u) \cdot n(g^{-1}u)) \\ &= e^{\rho-\nu}(a(g^{-1}u)) \cdot f(k(g^{-1}u)). \end{aligned}$$

We conclude that $\pi_{E,\nu}(g)$ acts on $\mathcal{C}^\infty(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_\nu)$ by the $G_{\mathbb{R}}$ -action on $G_{\mathbb{R}}/P_{\mathbb{R}} \simeq K_{\mathbb{R}}/M_{\mathbb{R}}$, $(g, u) \mapsto k(gu)$, and multiplication by a real analytic function

$$G_{\mathbb{R}} \times K_{\mathbb{R}} \rightarrow \mathbb{C}, \quad (g, u) \mapsto e^{\rho-\nu}(a(g^{-1}u)),$$

called the *factor of automorphy*.

The $K_{\mathbb{R}}/M_{\mathbb{R}}$ picture allows us to identify the space of $K_{\mathbb{R}}$ -finite vectors in $(\pi_{E,\nu}, \mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu))$. By Peter-Weyl Theorem, the space of $K_{\mathbb{R}} \times K_{\mathbb{R}}$ -finite vectors in $\mathcal{C}^\infty(K_{\mathbb{R}})$ is

$$\mathcal{C}^\infty(K_{\mathbb{R}})_{fini} \simeq \bigoplus_{i \in \hat{K}_{\mathbb{R}}} (\tau_i \boxtimes \tau_i^*, U_i \boxtimes U_i^*).$$

Therefore, the space of $K_{\mathbb{R}}$ -finite vectors

$$\mathcal{C}^\infty(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_\nu)_{fini} \simeq \bigoplus_{i \in \hat{K}_{\mathbb{R}}} U_i \otimes \text{Hom}_{M_{\mathbb{R}}}(U_i, E),$$

where $\text{Hom}_{M_{\mathbb{R}}}(U_i, E)$ denotes the space of $M_{\mathbb{R}}$ -equivariant linear maps $U_i \rightarrow E$ ($M_{\mathbb{R}}$ acts via $\tau_i|_{M_{\mathbb{R}}}$ on U_i and via τ on E) and $K_{\mathbb{R}}$ acts via τ_i on U_i and trivially on $\text{Hom}_{M_{\mathbb{R}}}(U_i, E)$. Assuming (τ, E) is irreducible, we can obtain a very crude estimate for the dimension of the i -isotypic component of $\mathcal{C}^\infty(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_\nu)$:

$$\begin{aligned} \dim \mathcal{C}^\infty(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_\nu)(i) &= \dim U_i \cdot \dim \text{Hom}_{M_{\mathbb{R}}}(U_i, E) \\ &= \dim U_i \cdot (\text{multiplicity of } (\tau, E) \text{ in } (\tau_i|_{M_{\mathbb{R}}}, U_i)) \\ &\leq (\dim U_i)^2. \end{aligned} \quad (9)$$

In particular, the principal series representations are admissible.

It is also true that the principal series representations have finite length. Moreover, if the representation (τ, E) of $M_{\mathbb{R}}$ is irreducible and $\nu \in \mathfrak{a}^*$ is “sufficiently generic”, the resulting

principal series representation $\pi_{E,\nu}$ of $G_{\mathbb{R}}$ is irreducible. The description of $\pi_{E,\nu}$ as a representation of $G_{\mathbb{R}}$ on vector space

$$\{f : K_{\mathbb{R}} \xrightarrow{\mathcal{C}^{\infty}} E; f(km) = \tau(m^{-1})f(k), \forall k \in K_{\mathbb{R}}, m \in M_{\mathbb{R}}\}$$

(in terms of factor of automorphy) shows that $\pi_{E,\nu}$ induces (continuous!) representations on various related spaces

$$\mathcal{C}^k(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_{\nu}), \quad 0 \leq k \leq \infty, k = -\infty, \omega,$$

and

$$L^p(K_{\mathbb{R}}/M_{\mathbb{R}}, \mathcal{E}_{\nu}), \quad 1 \leq p < \infty,$$

consisting of functions $f : K_{\mathbb{R}} \rightarrow E$ that are L^p with respect to the Haar measure on $K_{\mathbb{R}}$ (and satisfy $f(km) = \tau(m^{-1})f(k)$ for all $k \in K_{\mathbb{R}}, m \in M_{\mathbb{R}}$ as before). All of these representations have the same $K_{\mathbb{R}}$ -structure, i.e. their spaces of $K_{\mathbb{R}}$ -finite vectors are the same as (\mathfrak{g}, K) -modules. So all these representations are infinitesimally equivalent to each other.

11.4 Subrepresentation Theorem

Of particular interest is the following highly non-trivial result due to W.Casselman:

Theorem 107 (Subrepresentation Theorem). *For every irreducible Harish-Chandra module W , there exist an irreducible representation (τ, E) of $M_{\mathbb{R}}$ and a $\nu \in \mathfrak{a}^*$ with an embedding*

$$W \hookrightarrow \mathcal{C}^{\infty}(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{\nu})_{fini}.$$

Corollary 108. *Every admissible irreducible representation of $G_{\mathbb{R}}$ is infinitesimally equivalent to a subrepresentation of some principal series representation.*

The following corollary plays the key role in constructing globalizations of Harish-Chandra modules discussed in Subsection 7.5.

Corollary 109. *Every Harish-Chandra module can be realized as the underlying Harish-Chandra module of an admissible representation of finite length. In other words, every Harish-Chandra module has a globalization.*

The Subrepresentation Theorem replaces a much older result due to Harish-Chandra:

Theorem 110 (Subquotient Theorem). *Any irreducible Harish-Chandra module arises as a subquotient (i.e. quotient of a submodule) of the underlying Harish-Chandra module of some principal series representation.*

11.5 Unitary Principal Series Representations

In this subsection we construct the unitary principal series representations. Suppose that $\nu \in i\mathfrak{a}_{\mathbb{R}}^*$, then $e^\nu : A_{\mathbb{R}} \rightarrow \mathbb{C}$ is unitary (but $e^{\nu-\rho}$ is not). Fix an $M_{\mathbb{R}}$ -invariant positive definite inner product on E , then get a $G_{\mathbb{R}}$ -invariant pairing

$$\begin{aligned} \mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu) \times \mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_\nu) \\ \rightarrow \mathcal{C}^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{L}_{-2\rho}) \simeq \Omega^{\text{top}}(G_{\mathbb{R}}/P_{\mathbb{R}}) \rightarrow \mathbb{C}. \end{aligned}$$

In this composition, the first map is taking inner product on the fibers of \mathcal{E}_ν pointwise over $G_{\mathbb{R}}/P_{\mathbb{R}}$, the second map is the isomorphism from Lemma 105 and the last map is given by integration over $G_{\mathbb{R}}/P_{\mathbb{R}}$. This composition results in a $G_{\mathbb{R}}$ -invariant positive definite hermitian pairing. Complete the representation

spaces to get action $\pi_{E,\nu}$ on

$$L^2(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{\nu}) \quad \text{for } \underline{\nu \in i\mathfrak{a}_{\mathbb{R}}^*}.$$

This representation is denoted by

$$\text{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}}(\tau \boxtimes e^{\nu})$$

and is often called an *induced representation*, the shift by ρ is implicit. Thus, for $\nu \in i\mathfrak{a}_{\mathbb{R}}^*$, we get the *unitary principal series representations*. When ν is in \mathfrak{a}^* and not necessarily in $i\mathfrak{a}_{\mathbb{R}}^*$, we get the “non-unitary” principal series representations, or – much better – “not necessarily unitary” principal series representations, since some of these may be unitary for totally non-obvious reasons.

Now, let $\nu \in \mathfrak{a}^*$ and let (τ^*, E^*) be the representation of $M_{\mathbb{R}}$ dual to (τ, E) . Form a representation $(\tau_{-\nu}^*, E^*)$ of $P_{\mathbb{R}}$ and let $\mathcal{E}_{-\nu}^*$ be the corresponding $G_{\mathbb{R}}$ -equivariant vector bundle over $G_{\mathbb{R}}/P_{\mathbb{R}}$. Then the same argument as before proves that the bilinear pairing $E \times E^* \rightarrow \mathbb{C}$ induces a $G_{\mathbb{R}}$ -invariant non-degenerate bilinear pairing

$$\mathcal{C}^{\infty}(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{\nu}) \times \mathcal{C}^{\infty}(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{-\nu}^*) \rightarrow \mathbb{C}.$$

Completing with respect to appropriate topology yields representations $\pi_{E,\nu}$ on, for example,

$$\mathcal{C}^{\infty}(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{\nu}), \quad L^p(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{\nu}), \quad 1 < p < \infty,$$

with dual representations $\pi_{E^*, -\nu}$ on

$$\mathcal{C}^{-\infty}(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{-\nu}^*), \quad L^q(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{E}_{-\nu}^*), \quad p^{-1} + q^{-1} = 1,$$

where $\mathcal{C}^{-\infty}$ denotes the space of distributions.

12 Group Characters

12.1 Hilbert-Schmidt Operators

Note that if the vector space V is infinite dimensional, then taking the trace of a linear transformation $T \in \text{End}(V)$ is problematic. In this subsection we discuss Hilbert-Schmidt operators. These will be used in the next subsection to define trace class operators. Let V be a Hilbert space and fix an orthonormal basis $\{v_i\}$ of V .

Definition 111. *A bounded operator $T \in \text{End}(V)$ is a Hilbert-Schmidt operator if $\sum_i \|Tv_i\|^2 < \infty$.*

We will see shortly that this definition as well as the next one do not depend on the choice of basis. We denote the space of Hilbert-Schmidt operators by $\text{End}(V)_{HS}$.

Definition 112. *Let T_1 and T_2 be two Hilbert-Schmidt operators, then their Hilbert-Schmidt inner product is*

$$\langle T_1, T_2 \rangle_{HS} =_{\text{def}} \sum_i \langle T_1 v_i, T_2 v_i \rangle = \text{Tr}(T_1^* T_2).$$

This inner product induces the Hilbert-Schmidt norm

$$\|T\|_{HS}^2 = \langle T, T \rangle_{HS} = \sum_i \|Tv_i\|^2.$$

This inner product turns the space of Hilbert-Schmidt operators into a Hilbert space (in particular, it is complete).

Note that $\|Tv_i\|^2 = \sum_j |\langle Tv_i, v_j \rangle|^2$, hence we obtain the following expression for the Hilbert-Schmidt norm:

$$\|T\|_{HS}^2 = \sum_i \|Tv_i\|^2 = \sum_{ij} |\langle Tv_i, v_j \rangle|^2.$$

The last sum is the sum of squares of absolute values of matrix coefficients of T in basis $\{v_i\}$.

Lemma 113. *If $T \in \text{End}(V)_{HS}$, then its adjoint T^* also is a Hilbert-Schmidt operator with the same norm. Moreover, if $T_1, T_2 \in \text{End}(V)_{HS}$, then*

$$\langle T_1, T_2 \rangle_{HS} = \overline{\langle T_1^*, T_2^* \rangle_{HS}}. \quad (10)$$

Proof. We have:

$$\begin{aligned} \|T\|_{HS}^2 &= \sum_{ij} |\langle T v_i, v_j \rangle|^2 = \sum_{ij} |\langle v_i, T^* v_j \rangle|^2 \\ &= \sum_{ij} |\langle T^* v_j, v_i \rangle|^2 = \|T^*\|_{HS}^2. \end{aligned}$$

The second part follows from the polarization identity:

$$\begin{aligned} &4\langle T_1, T_2 \rangle_{HS} \\ &= \|T_1 + T_2\|_{HS}^2 - \|T_1 - T_2\|_{HS}^2 + i\|T_1 + iT_2\|_{HS}^2 - i\|T_1 - iT_2\|_{HS}^2 \\ &= \|T_1^* + T_2^*\|_{HS}^2 - \|T_1^* - T_2^*\|_{HS}^2 + i\|T_1^* - iT_2^*\|_{HS}^2 - i\|T_1^* + iT_2^*\|_{HS}^2 \\ &= 4\overline{\langle T_1^*, T_2^* \rangle_{HS}}. \end{aligned}$$

□

Lemma 114. *Let $T \in \text{End}(V)_{HS}$, $A \in \text{End}(V)$ with operator norm $\|A\|$. Then $AT, TA \in \text{End}(V)_{HS}$ and*

$$\|AT\|_{HS} \leq \|A\| \cdot \|T\|_{HS}, \quad \|TA\|_{HS} \leq \|A\| \cdot \|T\|_{HS}.$$

Remark 115. *This lemma can be restated as follows. The Hilbert-Schmidt operators form a two-sided ideal in the Banach algebra of bounded linear operators on V .*

Proof. We have:

$$\|AT\|_{HS}^2 = \sum_i \|ATv_i\|^2 \leq \|A\|^2 \sum_i \|Tv_i\|^2 = \|A\|^2 \cdot \|T\|_{HS}^2.$$

Finally, $TA \in \text{End}(V)_{HS}$ if and only if $(TA)^* \in \text{End}(V)_{HS}$ and

$$\begin{aligned} \|TA\|_{HS} &= \|(TA)^*\|_{HS} = \|A^*T^*\|_{HS} \\ &\leq \|A^*\| \cdot \|T^*\|_{HS} = \|A\| \cdot \|T\|_{HS}. \end{aligned}$$

□

Corollary 116. *For $U \in \text{Aut}(V)$ unitary, $\|UTU^{-1}\|_{HS} = \|T\|_{HS}$.*

Proof. Since U is unitary, $\|U\| = \|U^{-1}\| = 1$. By above lemma, $\|UTU^{-1}\|_{HS} \leq \|T\|_{HS}$. By symmetry, $\|T\|_{HS} \leq \|UTU^{-1}\|_{HS}$ and we have equality $\|UTU^{-1}\|_{HS} = \|T\|_{HS}$. □

Now, let $\{\tilde{v}_i\}$ be another orthonormal basis of V , then there exists a unique linear transformation U such that $U\tilde{v}_i = v_i$ for all i , then $U^{-1}v_i = \tilde{v}_i$ for all i and U is automatically unitary. We have:

$$\begin{aligned} \|T\|_{HS}^2 &= \|UTU^{-1}\|_{HS}^2 = \sum_i \|UTU^{-1}v_i\|^2 \\ &= \sum_i \|UT\tilde{v}_i\|^2 = \sum_i \|T\tilde{v}_i\|^2. \end{aligned}$$

This calculation shows that we can take any orthonormal basis of V without affecting the Hilbert-Schmidt norm or inner product.

12.2 Trace Class Operators

In this subsection we define trace class operators. As the name suggests, there is a well defined notion of trace for these operators.

Definition 117. A trace class operator on V is a finite linear combination of compositions $S_1 S_2$, where $S_1, S_2 \in \text{End}(V)_{HS}$. In other words,

$$\{\text{trace class operators}\} = (\text{End}(V)_{HS})^2$$

in the sense of ideals in the Banach algebra of bounded linear operators on V .

Let $\{v_i\}$ be an orthonormal basis of V and suppose that $T = S_1 S_2$ with $S_1, S_2 \in \text{End}(V)_{HS}$. Then

$$\langle S_2, S_1^* \rangle_{HS} = \sum_i \langle S_2 v_i, S_1^* v_i \rangle = \sum_i \langle S_1 S_2 v_i, v_i \rangle = \sum_i \langle T v_i, v_i \rangle$$

is absolutely convergent and does not depend on the choice of orthonormal basis $\{v_i\}$. We define $\text{Tr } T = \langle S_2, S_1^* \rangle_{HS}$ if $T = S_1 S_2$ and extend this definition by linearity to all trace class operators. Then $\text{Tr } T$ is well defined, i.e. independent of the decomposition of T into a linear combination of products of pairs of Hilbert-Schmidt operators and the choice of orthonormal basis. This definition is consistent with the notion of trace in finite dimensional case.

Lemma 118. If $T \in \text{End}(V)$ is of trace class and $A \in \text{End}(V)$ is bounded, then AT , TA are of trace class and $\text{Tr}(AT) = \text{Tr}(TA)$. In particular, if A has a bounded inverse, then ATA^{-1} is of trace class and $\text{Tr}(ATA^{-1}) = \text{Tr } T$.

Proof. By linearity, we may assume that $T = S_1 S_2$ with $S_1, S_2 \in \text{End}(V)_{HS}$. Then $AT = (AS_1)S_2$, $TA = S_1(S_2 A)$ with $AS_1, S_2 A \in \text{End}(V)_{HS}$, thus AT , TA are of trace class. Applying (10), we

obtain:

$$\begin{aligned}\mathrm{Tr}(AT) &= \langle S_2, (AS_1)^* \rangle_{HS} = \overline{\langle S_2^*, AS_1 \rangle_{HS}} = \overline{\langle A^* S_2^*, S_1 \rangle_{HS}} \\ &= \overline{\langle (S_2 A)^*, S_1 \rangle_{HS}} = \langle S_2 A, S_1^* \rangle_{HS} = \mathrm{Tr}(TA).\end{aligned}$$

□

Remark 119. *We have the following inclusions of two-sided ideals in the Banach algebra of bounded linear operators on V :*

$$\begin{aligned}\left\{ \begin{array}{l} \text{finite rank} \\ \text{operators} \end{array} \right\} &\subset \left\{ \begin{array}{l} \text{trace class} \\ \text{operators} \end{array} \right\} \\ &\subset \left\{ \begin{array}{l} \text{Hilbert-Schmidt} \\ \text{operators} \end{array} \right\} \subset \left\{ \begin{array}{l} \text{compact} \\ \text{operators} \end{array} \right\}.\end{aligned}$$

If V is infinite dimensional these inclusions are all proper.

12.3 The Definition of a Character

Let (π, V) be an admissible representation of $G_{\mathbb{R}}$ of finite length on a Hilbert space V (but not necessarily unitary). Fix a maximal compact subgroup $K_{\mathbb{R}} \subset G_{\mathbb{R}}$. We can change the inner product on V without changing the Hilbert space topology so that $K_{\mathbb{R}}$ acts unitarily. (First average the inner product over the $K_{\mathbb{R}}$ -action, and then argue using the uniform boundedness principle that the topology with respect to the new inner product is the same as before.) Then

$$V_{\text{fini}} = \bigoplus_{i \in \hat{K}_{\mathbb{R}}} V(i)$$

becomes an orthogonal decomposition. Take an orthonormal basis in each $V(i)$, put these bases together to get a (Hilbert

space) orthonormal basis $\{v_k\}$ of V . Fix a bi-invariant Haar measure dg on $G_{\mathbb{R}}$.

Theorem 120 (Harish-Chandra). *For each $f \in \mathcal{C}_c^\infty(G_{\mathbb{R}})$, the operator*

$$\pi(f) = \int_{G_{\mathbb{R}}} f(g) \pi(g) dg$$

is of trace class, and the linear functional

$$\Theta_\pi : f \mapsto \text{Tr } \pi(f)$$

is a conjugate-invariant distribution on $G_{\mathbb{R}}$.

Remark 121. *Note that $\pi(f)$ (and hence $\text{Tr } \pi(f)$) depends on the product $f(g) dg$ and not directly on f or dg . The expression $f(g) dg$ can be interpreted as a smooth compactly supported measure on $G_{\mathbb{R}}$. Thus the theorem asserts that Θ_π is a continuous linear functional on the space of smooth compactly supported measures on $G_{\mathbb{R}}$.*

Let $\theta_k : G_{\mathbb{R}} \rightarrow \mathbb{C}$ be the diagonal matrix coefficient functions $\langle \pi(g)v_k, v_k \rangle$. The proof shows that $\Theta_\pi = \sum_k \theta_k$ as distributions and the convergence is in the weak- topology (i.e. $\Theta_\pi(f) = \sum_k \int_{G_{\mathbb{R}}} \theta_k(g) \cdot f(g) dg$, for all $f \in \mathcal{C}_c^\infty(G_{\mathbb{R}})$). In other words, the distribution Θ_π extends the notion of the character of a finite dimensional representation. For this reason, Θ_π is called the Harish-Chandra character of (π, V) .*

We will use a formal notation of integration for the pairing between distributions and smooth functions with compact support and write

$$\text{Tr } \pi(f) = \int_{G_{\mathbb{R}}} \Theta_\pi(g) \cdot f(g) dg$$

for $\Theta_\pi(f)$.

Proof. Let $\Omega_{\mathfrak{k}} \in \mathcal{U}(\mathfrak{k})$ be the Casimir element of \mathfrak{k} . Pick a Cartan subalgebra \mathfrak{t} of \mathfrak{k} and a system of positive roots relative to \mathfrak{t} . Let

$$\rho_c = \frac{1}{2}(\text{sum of positive roots}) \in \mathfrak{t}^*.$$

For each $i \in \hat{K}_{\mathbb{R}}$, let $\mu_i \in \mathfrak{t}^*$ denote the highest weight of (τ_i, U_i) . Since (τ_i, U_i) is irreducible, $\Omega_{\mathfrak{k}}$ acts on U_i by multiplication by a scalar. The value of that scalar is easily determined by checking the action of $\Omega_{\mathfrak{k}}$ on the highest weight vector of U_i :

$$(\mu_i + \rho_c, \mu_i + \rho_c) - (\rho_c, \rho_c) = (\mu_i + 2\rho_c, \mu_i) \geq \|\mu_i\|^2.$$

Note that μ_i is a complete invariant of (τ_i, U_i) and ranges over a lattice intersected with certain cone (namely, the positive Weyl chamber). Therefore, for n sufficiently large,

$$\sum_{i \in \hat{K}_{\mathbb{R}}} (1 + \|\mu_i\|^2)^{-n} < \infty.$$

Recall that the Weyl dimension formula implies that $\dim U_i$ is expressible as a polynomial function of μ_i . We also know from (9) (at least for the principal series representations and hence for all of their subrepresentations) that

$$\dim V(i) \leq (\text{length of } (\pi, V)) \cdot (\dim U_i)^2,$$

which is bounded by a polynomial in $(1 + \|\mu_i\|^2)$. Note that, for $n > 0$,

$$\pi(1 + \Omega_{\mathfrak{k}})^{-n} : \bigoplus_{i \in \hat{K}_{\mathbb{R}}} V(i) \rightarrow \bigoplus_{i \in \hat{K}_{\mathbb{R}}} V(i)$$

extends to a bounded operator on V – in fact, a Hilbert-Schmidt operator – if n is sufficiently large. For $f \in \mathcal{C}_c^\infty(G_{\mathbb{R}})$, write

$$\pi(f) = \pi(r(1 + \Omega_{\mathfrak{k}})^{2n} f) \cdot \pi(1 + \Omega_{\mathfrak{k}})^{-n} \cdot \pi(1 + \Omega_{\mathfrak{k}})^{-n}. \quad (11)$$

Thus we have expressed $\pi(f)$ as a composition of two Hilbert-Schmidt operators, hence $\pi(f)$ is of trace class. This proves that $\text{Tr } \pi(f)$ is well-defined and equals $\sum_k \int_{G_{\mathbb{R}}} \theta_k \cdot f(g) dg$, where $\theta_k : G_{\mathbb{R}} \rightarrow \mathbb{C}$ are the diagonal matrix coefficient functions $\langle \pi(g)v_k, v_k \rangle$.

Note that

$$\|\pi(r(1 + \Omega_{\mathfrak{k}})^{2n}f)\| \leq C \cdot \sup |r(1 + \Omega_{\mathfrak{k}})^{2n}f|,$$

where the constant C depends only on the support of f (follows from the uniform boundedness principle). This means that in the factorization (11) of $\pi(f)$ one Hilbert-Schmidt operator is independent of f and the other has Hilbert-Schmidt norm bounded by the operator norm of $\pi(r(1 + \Omega_{\mathfrak{k}})^{2n}f)$. We conclude that the map $f \mapsto \text{Tr } \pi(f)$ is continuous in the topology of $\mathcal{C}_c^\infty(G_{\mathbb{R}})$, i.e. Θ_π is a distribution.

Finally, for $g \in G_{\mathbb{R}}$, we can write f conjugated by g as $l(g)r(g)f$, then we get

$$\text{Tr } \pi(l(g)r(g)f) = \text{Tr } (\pi(g^{-1})\pi(f)\pi(g)) = \text{Tr } \pi(f),$$

which proves that the distribution Θ_π is conjugation-invariant. \square

12.4 Properties of Characters

As before, let $\{v_k\}$ be a (Hilbert space) orthonormal basis of V obtained by combining orthonormal bases for all $V(i)$, $i \in \hat{K}_{\mathbb{R}}$. By Theorem 58, each diagonal matrix coefficient function $\theta_k(g) = \langle \pi(g)v_k, v_k \rangle$ is real analytic. We have already mentioned that $\Theta_\pi = \sum_k \theta_k$ in the sense of distributions. Note that, for each k , the Taylor series of θ_k at the identity element

(or at an arbitrary point of $K_{\mathbb{R}}$ when $G_{\mathbb{R}}$ is not connected) is completely determined by the structure of V_{fini} as a Harish-Chandra module. Thus we conclude:

Lemma 122. *The characters of any two infinitesimally equivalent representations coincide.*

Also observe that if

$$0 \rightarrow (\pi_1, V_1) \rightarrow (\pi, V) \rightarrow (\pi_2, V_2) \rightarrow 0$$

is a short exact sequence of representations of $G_{\mathbb{R}}$, then

$$\Theta_{\pi} = \Theta_{\pi_1} + \Theta_{\pi_2}$$

(even if the sequence does not split). Thus we can say that the character of a representation (π, V) is an invariant of the “infinitesimal semisimplification of (π, V) ”.

Theorem 123. *Let $\{(\pi_{\alpha}, V_{\alpha})\}_{\alpha \in \mathcal{A}}$ be a collection of irreducible admissible representations of $G_{\mathbb{R}}$, no two of which are infinitesimally equivalent. Then $\{\Theta_{\pi_{\alpha}}\}_{\alpha \in \mathcal{A}}$ is a linearly independent set.*

Corollary 124. *An irreducible admissible representation of $G_{\mathbb{R}}$ is uniquely determined by its character, up to infinitesimal equivalence.*

Corollary 125. *The character Θ_{π} completely determines the irreducible components of (π, V) , up to infinitesimal equivalence.*

If $\dim V < \infty$, (π, V) is the direct sum of its irreducible components, hence completely determined by Θ_{π} . If $\dim V = \infty$, the character Θ_{π} determines the irreducible components of

(π, V) , but cannot tell how (π, V) is built out of them. Hence Θ_π only determines the infinitesimal equivalence class of the semisimplification of (π, V) .

As usual, we assume that $G_{\mathbb{R}}$ is connected and let $\mathcal{ZU}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Recall the isomorphism from Proposition 18:

$$\mathcal{U}(\mathfrak{g}) \simeq \left\{ \begin{array}{l} \text{algebra of left-invariant} \\ \text{differential operators on } G_{\mathbb{R}} \end{array} \right\},$$

under this isomorphism,

$$\mathcal{ZU}(\mathfrak{g}) \simeq \left\{ \begin{array}{l} \text{algebra of bi-invariant} \\ \text{differential operators on } G_{\mathbb{R}} \end{array} \right\}.$$

Suppose that the representation (π, V) is irreducible, admissible. Then $\mathcal{ZU}(\mathfrak{g})$ acts on V by an infinitesimal character $\chi_\pi : \mathcal{ZU}(\mathfrak{g}) \rightarrow \mathbb{C}$. This means that, for all $z \in \mathcal{ZU}(\mathfrak{g})$ and all $v \in V^\infty$, $\pi(z)v = \chi_\pi(z)v$. In particular, for a diagonal matrix coefficient $\theta_k(g) = \langle \pi(g)v_k, v_k \rangle$, we have $r(z)\theta_k = \chi_\pi(z)\theta_k$. Since $\Theta_\pi = \sum_k \theta_k$, we obtain:

Corollary 126. *For all $z \in \mathcal{ZU}(\mathfrak{g})$, we have $r(z)\Theta_\pi = \chi_\pi(z)\Theta_\pi$. In other words, the character of an irreducible representation Θ_π is a conjugation-invariant eigendistribution.*

Remark 127. *This is a very important property of characters of irreducible representations. Other properties of characters – such as Harish-Chandra Regularity Theorem – are proved in a slightly more general setting of conjugation-invariant eigendistributions. But not every conjugation-invariant eigendistribution is the character of a representation.*

If (π, V) is admissible of finite length (but not necessarily irreducible), Θ_π is a finite sum of conjugation-invariant eigendistributions. This implies that Θ_π is annihilated by an ideal $I \subset \mathcal{ZU}(\mathfrak{g})$ of finite codimension.

12.5 Harish-Chandra Regularity Theorem

As before, we denote the complexification of $G_{\mathbb{R}}$ by G . For an element $g \in G$, the following three properties are equivalent:

- g acts semisimply in every finite-dimensional representation of G ;
- It does so for one faithful finite-dimensional representation;
- The operator $\text{Ad}(g)$ on \mathfrak{g} is semisimple.

We call such elements of G *semisimple*. Similarly, one can define *unipotent* elements of G . The unipotent elements constitute a proper algebraic subvariety in G and a real-analytic subvariety in $G_{\mathbb{R}}$.

For an element $g \in G$, we use the following notation for the centralizer of g :

$$Z_G(g) = \{h \in G; gh = hg\}.$$

We have the following analogue of the Jordan Canonical Form Theorem:

Theorem 128 (Borel). *Every $g \in G$ can be written uniquely in the form $g = g_{ss}g_u$, where $g_{ss} \in G$ is semisimple and $g_u \in G$ is unipotent, and $g_{ss}g_u = g_u g_{ss}$. Moreover, g_{ss} and g_u lie in the center of $Z_G(g)$ and*

$$Z_G(g) = Z_G(g_{ss}) \cap Z_G(g_u).$$

And the same is true for $G_{\mathbb{R}}$ in place of G .

We call an element $g \in G$ *regular* if $\dim Z_G(g)$ is minimal. An element $g \in G_{\mathbb{R}}$ is regular if it is regular as an element of G or, equivalently, if $\dim Z_{G_{\mathbb{R}}}(g)$ is minimal. Set

$$G_{rs} =_{\text{def}} \text{set of all regular semisimple elements in } G,$$

$$(G_{\mathbb{R}})_{rs} =_{\text{def}} G_{\mathbb{R}} \cap G_{rs}.$$

Now we give another description of the regular semisimple set G_{rs} . Define a function D_0 on G by

$$\det(\text{Ad}(g) + (\lambda - 1)) = \lambda^r D_0(g) + \lambda^{r+1} D_1(g) + \dots,$$

where r is the smallest power of λ such that its coefficient $D_0(g)$ is not identically zero. For example, suppose that $H \subset G$ is a Cartan subgroup, then, for $g \in H$,

$$\det(\text{Ad}(g) + (\lambda - 1)) = \lambda^r \prod_{\alpha \in \Phi} (e^{\alpha}(g) - 1 + \lambda),$$

with $r = \dim H$. Hence

$$D_0(g)|_H = \prod_{\alpha \in \Phi} (e^{\alpha}(g) - 1). \quad (12)$$

Then

$$G_{rs} = \{g \in G; D_0(g) \neq 0\},$$

$$(G_{\mathbb{R}})_{rs} = \{g \in G_{\mathbb{R}}; D_0(g) \neq 0\}. \quad (13)$$

In fact, $(G_{\mathbb{R}})_{rs}$ is a Zariski open conjugation-invariant subset of $G_{\mathbb{R}}$, its complement has measure zero.

We are almost ready to state the Harish-Chandra Regularity Theorem. We identify functions F and distributions obtained by integrating against these functions F :

$$F \quad \rightsquigarrow \quad f \mapsto \int_{G_{\mathbb{R}}} F(g) \cdot f(g) dg.$$

Note that the integral $\int_{G_{\mathbb{R}}} F(g) \cdot f(g) dg$ is finite for all test functions $f \in \mathcal{C}_c^\infty(G_{\mathbb{R}})$ if and only if F is *locally integrable*, i.e. $\int_C |F(g)| dg < \infty$ for all compact subsets $C \subset G_{\mathbb{R}}$. Not every distribution on $G_{\mathbb{R}}$ can be expressed in this form. For example, the delta distributions and their derivatives cannot be expressed as integrals against locally integrable functions.

Theorem 129 (Harish-Chandra Regularity Theorem). *Let Θ be a conjugation-invariant eigendistribution for $\mathcal{ZU}(\mathfrak{g})$. Then*

- a) $\Theta|_{(G_{\mathbb{R}})_{rs}}$ *is a real-analytic function;*
- b) *This real-analytic function – regarded as a measurable function on all of $G_{\mathbb{R}}$ – is locally integrable;*
- c) *This locally integrable function is the distribution Θ .*

In part (a), the restriction of Θ to $(G_{\mathbb{R}})_{rs}$ means that we evaluate the distribution Θ in smooth functions f with $\text{supp } f \subset (G_{\mathbb{R}})_{rs}$ only. Intuitively, Θ is a real-analytic function, possibly with singularities along the complement of $(G_{\mathbb{R}})_{rs}$; part (c) can be interpreted as “there are no delta functions or their derivatives hiding in $G_{\mathbb{R}} \setminus (G_{\mathbb{R}})_{rs}$ ”.

The proof of this theorem also shows that the set of Harish-Chandra characters attached to a single infinitesimal character is finite dimensional. It is easy to show that the principal series representations *have* infinitesimal characters, and we already know that they are admissible. We can conclude that the principal series representations have finite length.

12.6 Characters of Representations of $SU(1, 1)$

In this subsection we give without proofs formulas for characters of some representations of $SU(1, 1)$.

An element in $SL(2, \mathbb{R})$ or $SU(1, 1)$ is semisimple if and only if it is diagonalizable. By (12)-(13), an element g in $SL(2, \mathbb{R})$ or $SU(1, 1)$ is regular semisimple if it is diagonalizable and its eigenvalues are not ± 1 , i.e. g has distinct eigenvalues. There are two possible scenarios: the eigenvalues of g are real and complex with non-zero imaginary part.

If $g \in SL(2, \mathbb{R})$ has distinct real eigenvalues, then g is $SL(2, \mathbb{R})$ -conjugate to $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, for some $a \in \mathbb{R}^\times$, such elements are called *hyperbolic*. If $g \in SL(2, \mathbb{R})$ has eigenvalues in $\mathbb{C} \setminus \mathbb{R}$, then the eigenvalues must be of the form $e^{i\theta}$ and $e^{-i\theta}$, for some $\theta \in (0, \pi) \cup (\pi, 2\pi)$, and g is $SL(2, \mathbb{R})$ -conjugate to $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, such elements are called *elliptic*. It is easy to tell apart elliptic and hyperbolic elements:

$$\begin{aligned} g \in SL(2, \mathbb{R}) \text{ is elliptic} & \iff |\operatorname{Tr} g| < 2, \\ g \in SL(2, \mathbb{R}) \text{ is hyperbolic} & \iff |\operatorname{Tr} g| > 2. \end{aligned}$$

Similarly, if $g \in SU(1, 1)$ has distinct real eigenvalues, it is called hyperbolic, and g is $SU(1, 1)$ -conjugate to $\pm \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, for some $t \in \mathbb{R}^\times$. If $g \in SU(1, 1)$ has eigenvalues in $\mathbb{C} \setminus \mathbb{R}$, it is called elliptic, and g is $SU(1, 1)$ -conjugate to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, for

some $\theta \in (0, \pi) \cup (\pi, 2\pi)$. As before,

$$\begin{aligned} g \in SU(1, 1) \text{ is elliptic} &\iff |\operatorname{Tr} g| < 2, \\ g \in SU(1, 1) \text{ is hyperbolic} &\iff |\operatorname{Tr} g| > 2. \end{aligned}$$

Furthermore, elements

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$$

are conjugate to each other via $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SU(1, 1)$, but

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

are *not* $SU(1, 1)$ -conjugate. (And a similar statement holds for $SL(2, \mathbb{R})$.) Thus, a conjugation-invariant function on $SU(1, 1)_{rs}$, such as a character, is uniquely determined by its values on the following elements

$$k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi),$$

together with

$$\pm a_t, \quad \text{where} \quad a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t > 0.$$

We start with finite dimensional representations. In this case the character is well-defined as a function on all elements of $SU(1, 1)$ and there is no need for invoking distributions.

Proposition 130. *Let (π_d, F_d) be the irreducible finite dimensional representation of $SU(1, 1)$ of dimension $d + 1$.*

Then

$$\begin{aligned}\mathrm{Tr} \pi_d(k_\theta) &= \frac{e^{i(d+1)\theta} - e^{-i(d+1)\theta}}{e^{i\theta} - e^{-i\theta}}, \\ \mathrm{Tr} \pi_d(a_t) &= \frac{e^{(d+1)t} - e^{-(d+1)t}}{e^t - e^{-t}}, \\ \mathrm{Tr} \pi_d(-a_t) &= (-1)^{d+1} \frac{e^{(d+1)t} - e^{-(d+1)t}}{e^t - e^{-t}}.\end{aligned}$$

Proof. Since $k_\theta = \exp(i\theta H)$, by Proposition 26, k_θ acts on F_d with eigenvalues

$$e^{-id\theta}, \quad e^{i(2-d)\theta}, \quad e^{i(4-d)\theta}, \quad \dots, \quad e^{i(d-2)\theta}, \quad e^{id\theta}.$$

Hence we get a geometric series

$$\begin{aligned}\mathrm{Tr} \pi_d(k_\theta) &= e^{-id\theta} + e^{i(2-d)\theta} + e^{i(4-d)\theta} + \dots + e^{i(d-2)\theta} + e^{id\theta} \\ &= \frac{e^{i(d+1)\theta} - e^{-i(d+1)\theta}}{e^{i\theta} - e^{-i\theta}}.\end{aligned}$$

To find $\mathrm{Tr} \pi_d(a_t)$, we notice that finite dimensional representations of $SU(1, 1)$ extend to $SL(2, \mathbb{C})$ and a_t is $SL(2, \mathbb{C})$ -conjugate to $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp(tH)$. Hence

$$\begin{aligned}\mathrm{Tr} \pi_d(a_t) &= \mathrm{Tr} \pi_d(\exp(tH)) \\ &= e^{-dt} + e^{(2-d)t} + e^{(4-d)t} + \dots + e^{(d-2)t} + e^{dt} \\ &= \frac{e^{(d+1)t} - e^{-(d+1)t}}{e^t - e^{-t}}.\end{aligned}$$

Finally, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SU(1, 1)$ acts on F_d trivially if d is odd and by $-Id$ if d is even. Hence

$$\mathrm{Tr} \pi_d(-a_t) = (-1)^{d+1} \frac{e^{(d+1)t} - e^{-(d+1)t}}{e^t - e^{-t}}.$$

□

Next we consider the holomorphic and antiholomorphic discrete series representations as well as their limits.

Theorem 131. *Let*

$$(\pi_\chi, \mathcal{D}_l^-) \quad \text{and} \quad (\pi_\chi, \mathcal{D}_l^+), \quad l = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \dots,$$

be the holomorphic and antiholomorphic discrete series representations of $SU(1, 1)$ together with their limits constructed in Section 8. We denote by $\Theta_{\pi_\chi}^-$ and $\Theta_{\pi_\chi}^+$ their respective characters. Then

$$\Theta_{\pi_\chi}^-(k_\theta) = \frac{e^{i(2l+1)\theta}}{e^{i\theta} - e^{-i\theta}}, \quad \Theta_{\pi_\chi}^+(k_\theta) = -\frac{e^{-i(2l+1)\theta}}{e^{i\theta} - e^{-i\theta}},$$

$$\Theta_{\pi_\chi}^-(a_t) = \Theta_{\pi_\chi}^+(a_t) = \frac{e^{(2l+1)|t|}}{|e^t - e^{-t}|},$$

$$\Theta_{\pi_\chi}^-(-a_t) = \Theta_{\pi_\chi}^+(-a_t) = (-1)^{2l} \frac{e^{(2l+1)|t|}}{|e^t - e^{-t}|}.$$

Note that $k_\theta = \exp(i\theta H)$ acts on \mathcal{D}_l^- with weights

$$e^{2il\theta}, \quad e^{2i(l-1)\theta}, \quad e^{2i(l-2)\theta}, \quad e^{2i(l-3)\theta}, \dots$$

Hence we get a geometric series

$$\begin{aligned} \Theta_{\pi_\chi}^-(k_\theta) &= e^{2il\theta} + e^{2i(l-1)\theta} + e^{2i(l-2)\theta} + e^{2i(l-3)\theta} + \dots \\ &= \lim_{n \rightarrow \infty} \frac{e^{i(2l+1)\theta} - e^{i(2l+1-2n)\theta}}{e^{i\theta} - e^{-i\theta}}. \end{aligned}$$

Now, $e^{i(2l+1-2n)\theta}$ does not approach zero as a function, but $\lim_{n \rightarrow \infty} e^{i(2l+1-2n)\theta} = 0$ in the sense of distributions. In other

words, let \tilde{F}_n be a distribution on $SU(1, 1)_{rs}$

$$\tilde{F}_n(f) = \int_{SU(1,1)} F_n(g) \cdot f(g) dg, \quad f \in \mathcal{C}_c^\infty(SU(1, 1)_{rs}),$$

where

$$F_n(g) = \begin{cases} e^{i(2l+1-2n)\theta} & \text{if } g \text{ is } SU(1, 1)\text{-conjugate to } k_\theta; \\ 0 & \text{otherwise,} \end{cases}$$

then $\lim_{n \rightarrow \infty} \tilde{F}_n(f) = 0$ for all $f \in \mathcal{C}_c^\infty(SU(1, 1)_{rs})$. This assertion follows from the fact that the Fourier coefficients $\hat{\varphi}(n)$ of a smooth function $\varphi(\theta)$ on a circle satisfy $\lim_{n \rightarrow \pm\infty} \hat{\varphi}(n) = 0$. Similarly, one can compute $\Theta_{\pi_\chi}^+(k_\theta)$. Computing $\Theta_{\pi_\chi}^\mp(a_t)$ requires more work.

Finally, we consider the principal series representations.

Theorem 132. *Let (π_χ, V_χ) , $\chi = (l, \varepsilon)$, be the principal series representations of $SU(1, 1)$ constructed in Section 8. Then*

$$\begin{aligned} \Theta_{\pi_\chi}(k_\theta) &= 0, \\ \Theta_{\pi_\chi}(a_t) &= \frac{e^{(2l+1)t} + e^{-(2l+1)t}}{|e^t - e^{-t}|}, \\ \Theta_{\pi_\chi}(-a_t) &= (-1)^{2\varepsilon} \frac{e^{(2l+1)t} + e^{-(2l+1)t}}{|e^t - e^{-t}|}. \end{aligned}$$

Note that when $l = -1, -3/2, -2, -5/2, \dots$ and $l + \varepsilon \in \mathbb{Z}$, the character of the principal series representation (π_χ, V_χ) is the sum of characters of the holomorphic discrete series $(\pi_\chi, \mathcal{D}_l^-)$, antiholomorphic discrete series representation $(\pi_\chi, \mathcal{D}_l^+)$ and the finite dimensional representation $F_{|2(l+1)|}$, as it should be by (5).

Also, when $l = -1/2$ and $\varepsilon = 1/2$, the character of (π_χ, V_χ) is the sum of characters of the limits of the holomorphic and antiholomorphic discrete series $(\pi_\chi, \mathcal{D}_{-1/2}^-)$ and $(\pi_\chi, \mathcal{D}_{-1/2}^+)$, which agrees with (6).

References

- [1] M. Atiyah, *Characters of Semisimple Lie Groups*, Lecture notes (1976), reprinted in Collected Works of Michael Atiyah, Vol. 4, Springer, 1988, pp. 491-557.
- [2] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) 48 (1947), 568–640.
- [3] V. Bolton and W. Schmid, *Discrete Series*, Proc. of Symposia in Pure Math. 61 (1997), 83-113.
- [4] W. Casselman, *Jacquet modules for real reductive groups*, Proc. of the International Congress of Mathematics, Helsinki, 1980, 557-563.
- [5] W. Casselman, *Canonical Extensions of Harish-Chandra modules to representations of G* , Can. Jour. Math. 41 (1989), 385-438.
- [6] Harish-Chandra, *On some applications of the universal enveloping algebra of a semisimple Lie algebra*, Trans. Amer. Math. Soc. 70 (1951), 28-96.
- [7] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space I* , Trans. Amer. Math. Soc. 75 (1953), 185-243.

- [8] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
- [9] R. Howe, E. C. Tan, *Non-Abelian Harmonic Analysis: Applications of $SL(2, \mathbb{R})$* , Springer, 1992.
- [10] M. Kashiwara and W. Schmid, *Quasi-equivariant \mathcal{D} -modules, equivariant derived category, and representations of reductive Lie groups*, Lie Theory and Geometry, in Honor of Bertram Kostant, Progress in Mathematics, vol. 123, Birkhäuser, Boston, 1994, pp. 457-488.
- [11] A. A. Kirillov, Jr., *An Introduction to Lie Groups and Lie Algebras*, Cambridge Univ Press, 2008.
- [12] A. W. Knap, *Representation theory of semisimple groups. An overview based on examples*, reprint of the 1986 original, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001.
- [13] A. Knap, *Lie Groups Beyond an Introduction*, Progress in Mathematics, vol. 140, Birkhäuser, 2002.
- [14] S. Lang, *$SL(2, \mathbb{R})$* , Graduate Texts in Mathematics, vol. 105, Springer-Verlag, New York, 1985.
- [15] W. Schmid, M. Libine, *Geometric methods in representation theory* in S.Gutt, J.Rawnsley, D.Sternheimer (Eds.), Poisson Geometry, Deformation Quantisation and Group Representations, London Math Society Lecture Note, vol. 323, Cambridge Univ. Press 2005, pp. 273-323.

- [16] W. Soergel, *An irreducible not admissible Banach representation of $SL(2, \mathbb{R})$* , Proc. Amer. Math. Soc, 104 (1988), 1322-1324.
- [17] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.
- [18] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer, 1984.
- [19] V. S. Varadarajan, *An introduction to harmonic analysis on semisimple Lie groups*, corrected reprint of the 1989 original, Cambridge Studies in Advanced Mathematics 16, Cambridge University Press, Cambridge, 1999.
- [20] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, translated from the Russian by V.N.Singh, Translations of Mathematical Monographs, Vol. 22, American Math Society, Providence, RI 1968.
- [21] N. R. Wallach, *Asymptotic expansion of generalized matrix entries of real reductive groups* in: Lie Group Representations I, Lecture Notes in Mathematics, vol. 1024, Springer, 1983, p. 287-369.
- [22] N. R. Wallach, *Real reductive groups I*, Pure and Applied Mathematics 132, Academic Press, Boston, MA, 1988.
- [23] D. A. Vogan, *Representations of Real Reductive groups*, Progress in Math, vol. 15, Birkhäuser, Boston, 1981.
- [24] G. Warner, *Harmonic Analysis on Semi-Simple Lie Groups*, vols. I and II, Springer-Verlag, 1972.